Stabilization of the Gas Flow in Star-Shaped Networks by Feedback Controls with Varying Delay

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Abstract. We consider the subcritical gas flow through star-shaped pipe networks. The gas flow is modeled by the isothermal Euler equations with friction. We stabilize the isothermal Euler equations locally around a given stationary state on a finite time interval. For the stabilization we apply boundary feedback controls with time-varying delays. The delays are given by C^1-functions with bounded derivatives. In order to analyze the system evolution, we introduce an L^2-Lyapunov function with delay terms. The boundary controls guarantee the exponential decay of the Lyapunov function with time.

Keywords: boundary feedback stabilization, Euler equations, gas network, Lyapunov function, star-shaped network, time-varying delay

1 Introduction

Recently, there has been intense research on the system dynamics in gas networks (see e.g. [1, 2, 5, 7, 8, 10, 13]). Due to the pipe wall friction, there is a loss of pressure along the pipe. A common model for the gas flow in pipes is the isothermal Euler equations with friction, a 2×2 PDE system of balance laws (see (1)). We study the isothermal Euler equations on a star-shaped network of N ($N \geq 2$) pipes that meet at a central node ω. The flow at the node ω is governed by the continuity of the density and conservation of mass (see (5)).
Our main result, stated in Theorem 1, is a method to stabilize the gas flow locally around a given stationary subcritical state on a finite time interval. To do so, we use boundary feedback controls with varying delays at the pipe ends

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which are not at the node ω (see (16)). In order to measure the system evolution, we introduce a network Lyapunov function (see (22)) which is the sum of a weighted and squared L^2-norm (see (20)) and a delay term (see (21)) for each pipe. The feedback controls guarantee the exponential decay of the Lyapunov function with time (see (36)) and, hence, the exponential stability of the system. In contrast to a previous work which studies only the case of *constant* delays (see [7]), the novelty of this paper is that we consider *nonconstant*, i.e. time-dependent, delays. This is very important for the daily operation of real gas networks. E.g., in the control of such networks via electrical and mechanical systems, nonconstant time delays often appear (see [3]).
This paper is structured as follows: In Sect. 2 we give the network notation, consider the isothermal Euler equations in terms of the physical and characteristic variables and present the coupling conditions at the node ω. In Sect. 3 stationary and nonstationary states are studied. The stabilization method, i.e. the feedback controls, the Lyapunov function and the exponential stability result (Theorem 1), are stated in Sect. 4. In Sect. 5 we prove Theorem 1.
The weighted and squared L^2-norm from (20) for the Euler equations has first been presented in [8]. It is an extension of the Lyapunov function introduced in [4]. Delay terms of the form (21) have been presented in [12] for the time-delayed stabilization of the wave equation. Related questions of the stabilization of the wave equation are e.g. studied in [6, 9, 14].

2 Gas Flow in a Star-Shaped Pipe Network

In this section we consider the gas flow in a star-shaped pipe network. First, we give the network notation. Then, we present the isothermal Euler equations in terms of the physical variables (see (1)) and in terms of the characteristic variables (see (3)). The coupling conditions at the central node of the network are stated in (5).

2.1 Network Notation

We consider a star-shaped network of N ($N \geq 2$) cylindrical pipes with the same diameter $\delta > 0$ that meet at a central node ω. We define the index set $I = \{1, ..., N\}$ and number the pipes from *pipe* 1 to *pipe* N. Variables referring to pipe i ($i \in I$) are denoted with a superscript (i). We model the pipes by a one-dimensional space model and parameterize the length $L^{(i)} > 0$ of pipe i by the space interval $[0,\, L^{(i)}]$ such that the end $x = 0$ is at the node ω. We consider the system on a finite time interval $[0,\, T]$ with $T > 0$.

2.2 Isothermal Euler Equations in Physical Variables

A common model for the system dynamics in gas pipes is the isothermal Euler equations with friction, a hyperbolic 2×2 system of balance laws (see [1, 2, 10, 13]): For pipe i ($i \in I$) they have the following form on $[0,\, T] \times [0,\, L^{(i)}]$:

$$\begin{cases} \partial_t\, \rho^{(i)}(t,x) \,+\, \partial_x\, q^{(i)}(t,x) = 0\;, \\ \partial_t\, q^{(i)}(t,x) \,+\, \partial_x \left(\frac{(q^{(i)}(t,x))^2}{\rho^{(i)}(t,x)} + a^2 \rho^{(i)}(t,x) \right) = -\frac{\theta}{2}\, \frac{q^{(i)}(t,x)\, |q^{(i)}(t,x)|}{\rho^{(i)}(t,x)} \end{cases} \tag{1}$$

where $\rho^{(i)}(t,x) > 0$ is the density of the gas and $q^{(i)}(t,x) \neq 0$ the mass flux. The sign of $q^{(i)}$ depends on the direction of the gas flow. It is positive if the gas in pipe i flows away from the node ω. The constant $a > 0$ is the sonic speed in the gas and the constant $\theta = \nu/\delta$ is the quotient of the friction factor $\nu > 0$ and the pipe diameter $\delta > 0$. The first equation in (1) states the conservation of mass and the second equation is the momentum equation. In this paper we study *subsonic* or *subcritical* C^1-states, i.e. C^1-states with $|q^{(i)}|/\rho^{(i)} < a$ for all $i \in I$. The equations (1) have the same form for each pipe. However, our calculations would also be true if we had different pressure laws on different pipes.

2.3 Isothermal Euler Equations in Characteristic Variables

The equations (1) can be transformed to the Riemann invariants (characteristic variables)

$$R_{\pm}^{(i)} \;=\; -q^{(i)}/\rho^{(i)} \mp a\, \ln(\rho^{(i)})\;. \tag{2}$$

For the calculation of the Riemann invariants see [5, 8]. In terms of $R_{\pm}^{(i)}$ the system (1) has the form

$$\partial_t \begin{pmatrix} R_{+}^{(i)} \\ R_{-}^{(i)} \end{pmatrix} + \begin{pmatrix} \lambda_{+}^{(i)} & 0 \\ 0 & \lambda_{-}^{(i)} \end{pmatrix} \partial_x \begin{pmatrix} R_{+}^{(i)} \\ R_{-}^{(i)} \end{pmatrix} = -\frac{\theta}{8}\, (R_{+}^{(i)} + R_{-}^{(i)})\, |R_{+}^{(i)} + R_{-}^{(i)}| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3}$$

with the eigenvalues

$$\lambda_{\pm}^{(i)} \;=\; \frac{q^{(i)}}{\rho^{(i)}} \pm a \;=\; -\frac{R_{+}^{(i)} + R_{-}^{(i)}}{2} \pm a\;. \tag{4}$$

In the subsonic case, $\lambda_{+}^{(i)}$ is strictly positive and $\lambda_{-}^{(i)}$ strictly negative.

2.4 Coupling Conditions

At the node ω, i.e. at the ends $x = 0$ of the pipes, we have the following coupling conditions in terms of the physical variables ($t \in [0,\, T]$) (see [1, 11]):

$$\rho^{(1)}(t,0) \,=\, \rho^{(i)}(t,0) \quad (i \in I\backslash\{1\}) \quad \text{and} \quad \sum_{i \in I} q^{(i)}(t,0) \,=\, 0\;. \tag{5}$$

The first equation in (5) says that the density is continuous at the node ω. Due to the parameterization of the pipes, the second equation in (5) means that the total ingoing mass flux is equal to the total outgoing mass flux at ω. The conditions (5) can equivalently be stated in terms of $R_{\pm}^{(i)}$ as ($t \in [0,\, T]$)

$$\left(R_+^{(1)}(t,0), ..., R_+^{(N)}(t,0)\right) \;=\; \left(R_-^{(1)}(t,0), ..., R_-^{(N)}(t,0)\right) A_\omega \qquad (6)$$

with the orthogonal, symmetric $(N \times N)$-matrix $A_\omega = (a_{kl})_{k,l=1}^N$ with the entries

$$a_{kk} = (N-2)/N \quad (k \in I) \quad \text{and} \quad a_{kl} = -2/N \quad (k,l \in I, k \neq l)\ . \qquad (7)$$

3 Stationary and Nonstationary States

3.1 Stationary States

The existence and behavior of stationary solutions of the isothermal Euler equations, i.e. solutions which do not explicitly depend on the time t, is studied in [5, 8]. We denote the stationary variables as $\bar{\rho}(x)$, $\bar{q}(x)$, $\bar{R}_{\pm}(x)$ and $\bar{\lambda}_{\pm}(x)$. In [5, 8] it is shown that $\bar{q}(x)$ is constant along a pipe and $\bar{\rho}(x)$ is strictly monotonically decreasing in the direction of the gas flow. Furthermore, a stationary subsonic C^1-solution of the isothermal Euler equations exists on the whole pipe if the pipe length is shorter than a critical length (see (8)). The critical length depends on the inflow density, the mass flux, the friction factor and the pipe diameter. For typical high-pressure gas pipes the critical length is around 180km (see [5]).
For the system (1) on a star-shaped network with the conditions (5), the existence and behavior of stationary subsonic C^1-solutions is in detail discussed in [7]. In the following we summarize the main results from [7]: The stationary mass fluxes $\bar{q}^{(i)} \neq 0$ are constant and have to satisfy $\sum_{i \in I} \bar{q}^{(i)} = 0$. At the node ω, we have a constant density $\bar{\rho}_\omega$ with $\bar{\rho}^{(i)}(0) = \bar{\rho}_\omega$ and $|\bar{q}^{(i)}|/\bar{\rho}_\omega < a$ for all $i \in I$. Furthermore, the lengths of the pipes with positive mass flux, i.e. with $\bar{q}^{(i)} > 0$, have to satisfy

$$L^{(i)} \;<\; \frac{1}{\theta}\left(a^2 \frac{\bar{\rho}_\omega^2}{(\bar{q}^{(i)})^2} - 1 + 2\ln\left(\frac{\bar{q}^{(i)}}{a\bar{\rho}_\omega}\right)\right)\ . \qquad (8)$$

3.2 Nonstationary States

Assume that on the star-shaped network we have a given stationary subsonic state $(\bar{\rho}^{(i)}(x), \bar{q}^{(i)})$ with the corresponding Riemann invariants $(\bar{R}_+^{(i)}(x), \bar{R}_-^{(i)}(x)) \in (C^1([0,\, L^{(i)}]))^2$ and the eigenvalues $\bar{\lambda}^{(i)}(x)$ $(i \in I)$ (see (2), (4)). We define the numbers (see (4))

$$\sigma^{(i)} \;=\; \mathrm{sign}(\bar{q}^{(i)}) \;=\; -\mathrm{sign}(\bar{R}_+^{(i)} + \bar{R}_-^{(i)}) \in \{-1, 1\}\ . \qquad (9)$$

Now we consider a nonstationary state $(\bar{R}_+^{(i)}(x) + r_+^{(i)}(t,x), \bar{R}_-^{(i)}(x) + r_-^{(i)}(t,x))$ on $[0,\, T] \times [0,\, L^{(i)}]$ in a local neighborhood of $(\bar{R}_+^{(i)}, \bar{R}_-^{(i)})$. That is we assume the given stationary state $\bar{R}_{\pm}^{(i)}(x)$ to be slightly perturbed by $r_{\pm}^{(i)}(t,x)$ where the

C^1-norms $||r_\pm^{(i)}||_{C^1([0,\,T]\times[0,\,L^{(i)}])}$ are small. In particular, we suppose that the mass flux directions of the nonstationary state are the same as of the stationary state, i.e. $(i \in I)$ (see (9))

$$\sigma^{(i)} \;=\; -\mathrm{sign}(\bar{R}_+^{(i)} + \bar{R}_-^{(i)} + r_+^{(i)} + r_-^{(i)})\;. \tag{10}$$

The stabilization method presented in Sect. 4 guarantees that the absolute values of $r_\pm^{(i)}$ are small enough (see (24), (35)), such that (10) holds and such that the direction of the mass fluxes does not change during the stabilization process. From (3) we obtain the following quasilinear system for $r_\pm^{(i)}(t,x)$ $(i \in I)$

$$\begin{cases} \partial_t\, r_+^{(i)} + (\bar{\lambda}_+^{(i)} - \frac{r_+^{(i)}+r_-^{(i)}}{2})\,\partial_x\, r_+^{(i)} = (r_+^{(i)} + r_-^{(i)})(-K_+^{(i)} + \sigma^{(i)}\frac{\theta}{8}(r_+^{(i)} + r_-^{(i)}))\;, \\ \partial_t\, r_-^{(i)} + (\bar{\lambda}_-^{(i)} - \frac{r_+^{(i)}+r_-^{(i)}}{2})\,\partial_x\, r_-^{(i)} = (r_+^{(i)} + r_-^{(i)})(-K_-^{(i)} + \sigma^{(i)}\frac{\theta}{8}(r_+^{(i)} + r_-^{(i)})) \end{cases} \tag{11}$$

on $[0,\,T] \times [0,\,L^{(i)}]$ with the strictly positive C^1-functions

$$K_\pm^{(i)}(x) \;=\; \frac{\theta}{8}\,|\bar{R}_+^{(i)}(x) + \bar{R}_-^{(i)}(x)|\,\frac{4a \mp (\bar{R}_+^{(i)}(x) + \bar{R}_-^{(i)}(x))}{2a \mp (\bar{R}_+^{(i)}(x) + \bar{R}_-^{(i)}(x))} \;>\; 0\;. \tag{12}$$

The linearity of the equation (6) implies that for $r_\pm^{(i)}(t,0)$ at the node ω we have the equation $(t \in [0,\,T])$

$$\left(r_+^{(1)}(t,0), ..., r_+^{(N)}(t,0)\right) \;=\; \left(r_-^{(1)}(t,0), ..., r_-^{(N)}(t,0)\right) A_\omega \tag{13}$$

with the matrix A_ω as in (7).

4 Feedback Stabilization with Time-Varying Delay

In this section we present a method for the stabilization of the system (11) on $[0,\,T] \times [0,\,L^{(i)}]$ with time-delayed feedbacks. The corresponding boundary controls are given in (16). In order to measure the system evolution, we define the Lyapunov function $\mathcal{F}(t)$ in (22) with the weighted and squared L^2-norms $\mathcal{E}^{(i)}(t)$ in (20) and the delay terms $\mathcal{D}^{(i)}(t)$ in (21). Theorem 1 states the existence of a unique C^1-solution of (11) with small C^1-norm (see (35)) for which $\mathcal{F}(t)$ decays exponentially with time (see (36)). Hence, Theorem 1 gives the exponential stability of the system (11) around $(r_+^{(i)}, r_-^{(i)}) = (0,0)$.

4.1 Boundary Feedback Controls with Time-Varying Delay

Let a finite time $T > 0$ and a stationary subsonic state $(\bar{R}_+^{(i)}(x), \bar{R}_-^{(i)}(x)) \in (C^1([0,\,L^{(i)}]))^2$ on the star-shaped network be given with the eigenvalues $\bar{\lambda}_\pm^{(i)}(x)$ as in (4) $(i \in I)$. Define the numbers $\sigma^{(i)} \in \{-1,1\}$ as in (9) and the functions

$K_\pm^{(i)}(x) \in C^1([0,\, L^{(i)}])$ as in (12). Let functions $\tau^{(i)}(t) \in C^1([0,\, T])$ $(i \in I)$ be given that satisfy $(t \in [0,\, T])$:

$$0 \,<\, \tau^{(i)}(t) \,<\, \frac{T}{2} \quad \text{and} \quad |\frac{d}{dt}\tau^{(i)}(t)| \,<\, 1\,. \tag{14}$$

Define the constants

$$\overline{\tau}^{(i)} = \max_{t\in[0,\,T]} \tau^{(i)}(t), \quad \hat{\tau}^{(i)} = \max_{t\in[0,\,T]} |\frac{d}{dt}\tau^{(i)}(t)| \quad \text{and} \quad \tau_{max} = \max_{i\in I}\left\{\overline{\tau}^{(i)}\right\}\,. \tag{15}$$

For a nonstationary state $\bar{R}_\pm^{(i)}(x) + r_\pm^{(i)}(t,x)$ on $[0,\, T] \times [0,\, L^{(i)}]$ we consider the system (11) with the condition (13) with the matrix A_ω as in (7). At the end $x = L^{(i)}$ of pipe i $(i \in I)$ we apply the controls

$$r_-^{(i)}(t, L^{(i)}) \;=\; \begin{cases} \vartheta^{(i)}(t) & (t \in [0,\, \overline{\tau}^{(i)}])\,, \\ (1-\varsigma^{(i)}(t))\,\vartheta^{(i)}(t) + \varsigma^{(i)}(t)\,k^{(i)}\,r_+^{(i)}(t-\tau^{(i)}(t), L^{(i)}) & (t \in (\overline{\tau}^{(i)},\, 2\overline{\tau}^{(i)}])\,, \\ k^{(i)}\,r_+^{(i)}(t-\tau^{(i)}(t), L^{(i)}) & (t \in (2\overline{\tau}^{(i)},\, T]) \end{cases} \tag{16}$$

with feedback constants $k^{(i)} \in (-1, 1)$ and functions $\vartheta^{(i)}(t) \in C^1([0,\, 2\overline{\tau}^{(i)}])$ and $\varsigma^{(i)}(t) \in C^1([\overline{\tau}^{(i)},\, 2\overline{\tau}^{(i)}])$ that have the following properties:

$$\varsigma^{(i)}(\overline{\tau}^{(i)}) \;=\; \frac{d}{dt}\varsigma^{(i)}(\overline{\tau}^{(i)}) \;=\; \frac{d}{dt}\varsigma^{(i)}(2\overline{\tau}^{(i)}) \;=\; 0 \quad \text{and} \quad \varsigma^{(i)}(2\overline{\tau}^{(i)}) \;=\; 1\,. \tag{17}$$

4.2 Network Lyapunov Function with Delay Terms

In order to define the network Lyapunov function, we define the numbers $(i \in I)$

$$\mu^{(i)} \;=\; \left(\int_0^{L^{(i)}} \frac{1}{\bar{\lambda}_+^{(i)}(x)} + \frac{1}{|\bar{\lambda}_-^{(i)}(x)|}\, dx\right)^{-1} \;>\; 0 \tag{18}$$

and the functions $(i \in I,\, x \in [0,\, L^{(i)}])$

$$h_\pm^{(i)}(x) \;=\; \exp\left(-\mu^{(i)} \int_0^x \frac{1}{\bar{\lambda}_\pm^{(i)}(\xi)}\, d\xi\right) \;>\; 0\,. \tag{19}$$

For constants $A_\pm^{(i)} \,>\, 0$ we define the weighted and squared L^2-norms $(i \in I,$ $t \in [0,\, T])$

$$\mathcal{E}^{(i)}(t) \;=\; \int_0^{L^{(i)}} \frac{A_+^{(i)}}{\bar{\lambda}_+^{(i)}(x)}\, h_+^{(i)}(x)\,(r_+^{(i)}(t,x))^2 \;+\; \frac{A_-^{(i)}}{|\bar{\lambda}_-^{(i)}(x)|}\, h_-^{(i)}(x)\,(r_-^{(i)}(t,x))^2\, dx \tag{20}$$

and the delay terms $(i \in I,\, t \in [\overline{\tau}^{(i)},\, T])$

$$\mathcal{D}^{(i)}(t) \;=\; \int_0^{\tau^{(i)}(t)} A_+^{(i)}\, h_+^{(i)}(L^{(i)})\, \exp(-\mu^{(i)} s)\,(r_+^{(i)}(t-s, L^{(i)}))^2\, ds\,. \tag{21}$$

The network Lyapunov function $\mathcal{F}(t)$ is defined as ($t \in [\tau_{max},\, T]$)

$$\mathcal{F}(t) \;=\; \sum_{i\in I} \mathcal{E}^{(i)}(t) \,+\, \mathcal{D}^{(i)}(t)\;. \tag{22}$$

Constants of the form $\mu^{(i)}$ in (18) and functions of the form $h_\pm^{(i)}(x)$ and $\mathcal{E}^{(i)}(t)$ in (19) and (20) have been introduced in [8]. Delay terms of the form (21) have been presented in [12].

4.3 Main Result: Exponential Stability

Theorem 1 states the existence of a unique C^1-solution of the system (11) on $[0,\, T] \times [0,\, L^{(i)}]$ ($i \in I$) with the boundary conditions (13) and (16) and initial data of the form ($i \in I$, $x \in [0,\, L^{(i)}]$)

$$(r_+^{(i)}(0,x), r_-^{(i)}(0,x)) \;=\; (\varphi_+^{(i)}(x), \varphi_-^{(i)}(x)) \tag{23}$$

with functions $\varphi_\pm^{(i)} \in C^1([0,\, L^{(i)}])$. For this solution, the function $\mathcal{F}(t)$ decays exponentially on $[2\tau_{max},\, T]$ (see (36)). The decay rate is $\eta = \min_{i\in I} \alpha^{(i)}\beta^{(i)}\mu^{(i)}$ with numbers $\alpha^{(i)} \in (0,\, 1)$ and $\beta^{(i)} \in (0,\, 1)$. The C^1-norm of the solution $r_\pm^{(i)}(t,x)$ is bounded by a constant $\varepsilon_1 > 0$ (see (35)) which has to satisfy ($i \in I$)

$$\varepsilon_1 < \min_{x\in[0,\, L^{(i)}]} |\bar\lambda_\pm^{(i)}(x)|\;, \quad 2\varepsilon_1 < \min_{x\in[0,\, L^{(i)}]} |\bar R_+^{(i)}(x) + \bar R_-^{(i)}(x)| \tag{24}$$

and

$$\varepsilon_1 \left(\frac{\theta}{4} + \frac{1}{2}\right) \left(3 + \max\left\{ \exp(1) \frac{A_-^{(i)}}{A_+^{(i)}} \frac{\bar\lambda_+^{(i)}(L^{(i)})}{|\bar\lambda_-^{(i)}(L^{(i)})|}, \frac{A_+^{(i)}}{A_-^{(i)}} \frac{|\bar\lambda_-^{(i)}(0)|}{\bar\lambda_+^{(i)}(0)} \right\}\right) < (1-\beta^{(i)})\alpha^{(i)}\mu^{(i)}. \tag{25}$$

The C^1-norms of the initial data $\varphi_\pm^{(i)}$ and the functions $\vartheta^{(i)}$ in the controls (16) have to be sufficiently small. More precisely, there exists a number $\varepsilon_2 \in (0,\, \varepsilon_1]$ such that the following inequalities have to hold:

$$||\varphi_\pm^{(i)}||_{C^1([0,\, L^{(i)}])} \;\le\; \varepsilon_2 \tag{26}$$

and

$$||\vartheta^{(i)}||_{C^1([0,\, \overline{\tau}^{(i)}])} \;\le\; \varepsilon_2\;, \quad ||(1-\varsigma^{(i)})\vartheta^{(i)}||_{C^1([\overline{\tau}^{(i)},\, 2\overline{\tau}^{(i)}])} \;\le\; \frac{\varepsilon_2}{2}\;. \tag{27}$$

Note that the second inequality in (27) holds for any $\varsigma^{(i)}$ if the C^1-norm of $\vartheta^{(i)}$ on $[\overline{\tau}^{(i)},\, 2\overline{\tau}^{(i)}]$ is small enough. For Theorem 1 we define the positive real numbers ($i \in I$)

$$U_\pm^{(i)} \;=\; \max_{x\in[0,\, L^{(i)}]} \left|\frac{\bar\lambda_\pm^{(i)}(x)}{\bar\lambda_\mp^{(i)}(x)}\right| \frac{K_\mp^{(i)}(x)}{K_\pm^{(i)}(x)} \;>\; 0 \tag{28}$$

and

$$V_\pm^{(i)} \;=\; \min_{x\in[0,\, L^{(i)}]} \left|\frac{\bar\lambda_\pm^{(i)}(x)}{\bar\lambda_\mp^{(i)}(x)}\right| \frac{K_\mp^{(i)}(x)}{K_\pm^{(i)}(x)} \left(1 + \frac{(1-\alpha^{(i)})\mu^{(i)}}{K_\mp^{(i)}(x)}\right) \;>\; 0\;. \tag{29}$$

Theorem 1. *Consider a star-shaped network of pipes as described in Sect. 2.1. Let a finite time $T > 0$ and functions $\tau^{(i)}(t) \in C^1([0,\,T])$ with (14) be given ($i \in I$). Define the constants $\overline{\tau}^{(i)}$, $\hat{\tau}^{(i)}$ and τ_{max} as in (15). Let real numbers $\alpha^{(i)} \in (0,\,1)$, $\beta^{(i)} \in (0,\,1)$ and a stationary subsonic state $(\bar{R}_+^{(i)}(x), \bar{R}_-^{(i)}(x)) \in (C^1([0,\,L^{(i)}]))^2$ with the eigenvalues $\bar{\lambda}_\pm^{(i)}(x)$ as in (4) be given ($i \in I$) that satisfies the coupling conditions (6). Define the numbers $\sigma^{(i)} \in \{-1, 1\}$ and $\mu^{(i)} > 0$ as in (9) and (18) and the functions $K_\pm^{(i)} > 0$ and $h_\pm^{(i)} > 0$ as in (12) and (19). Define the numbers $U_\pm^{(i)} > 0$ and $V_\pm^{(i)} > 0$ as in (28) and (29) and assume that we have*

$$\exp(1)\, U_+^{(i)} \,\le\, V_+^{(i)} \quad \text{or} \quad \exp(1)\, U_-^{(i)} \,\le\, V_-^{(i)}\,. \tag{30}$$

Choose constants $A_\pm^{(i)} > 0$ that satisfy

$$A_+^{(i)} \,\le\, 1 \,\le\, A_-^{(i)} \tag{31}$$

and assume that we have

$$A_+^{(i)}/A_-^{(i)} \in [\exp(1)\, U_+^{(i)},\, V_+^{(i)}] \quad \text{or} \quad A_-^{(i)}/A_+^{(i)} \in [U_-^{(i)},\, \exp(-1)\, V_-^{(i)}]\,. \tag{32}$$

Choose a real number $\varepsilon_1 > 0$ that satisfies (24) and (25) for all $i \in I$. Then there exists $\varepsilon_2 \in (0,\,\varepsilon_1]$ such that the following statements hold:
Choose functions $\vartheta^{(i)}(t) \in C^1([0,\,2\overline{\tau}^{(i)}])$, $\varsigma^{(i)}(t) \in C^1([\overline{\tau}^{(i)},\,2\overline{\tau}^{(i)}])$ and $\varphi_\pm^{(i)}(x) \in C^1([0,\,L^{(i)}])$ ($i \in I$) that satisfy (17), (26) and (27) and such that the C^1-compatibility conditions are satisfied at the points $(t,x) = (0,0)$ and $(t,x) = (0, L^{(i)})$ (see Remark 1). Choose constants $k^{(i)} \in (-1,1)$ ($i \in I$) that satisfy

$$\exp(1)\, (k^{(i)})^2\, A_-^{(i)} \,\le\, A_+^{(i)}\, \exp(-\mu^{(i)}\overline{\tau}^{(i)})\, (1 - \hat{\tau}^{(i)}) \tag{33}$$

and

$$|k^{(i)}| \,\le\, \varepsilon_2/(8\varepsilon_1 ||\varsigma^{(i)}||_{C^1([\overline{\tau}^{(i)},\, 2\overline{\tau}^{(i)}])})\,. \tag{34}$$

Then the initial-boundary value problem (11), (13), (16), (23) has a unique solution $(r_+^{(i)}, r_-^{(i)}) \in (C^1([0,\,T] \times [0,\,L^{(i)}]))^2$ that satisfies

$$||r_\pm^{(i)}||_{C^1([0,\,T]\times[0,\,L^{(i)}])} \,\le\, \varepsilon_1\,. \tag{35}$$

For this solution define the functions $\mathcal{E}^{(i)}(t)$, $\mathcal{D}^{(i)}(t)$ and $\mathcal{F}(t)$ as in (20), (21) and (22). Then the Lyapunov function $\mathcal{F}(t)$ satisfies the following inequality with $\eta = \min_{i\in I} \alpha^{(i)}\beta^{(i)}\mu^{(i)}$:

$$\mathcal{F}(t) \,\le\, \mathcal{F}(2\tau_{max})\, \exp\left(-\eta(t - 2\tau_{max})\right) \quad \text{for} \quad t \in [2\tau_{max},\, T]\,. \tag{36}$$

Remark 1. The C^1-compatibility conditions guarantee that the initial data (23) and the boundary conditions (13) and (16) and their first derivatives fit together at the points $(t,x) = (0,0)$ and $(t,x) = (0, L^{(i)})$ ($i \in I$). They can be calculated from $\varphi_\pm^{(i)}$, $\vartheta_\pm^{(i)}$, A_ω and the equations (11) (see [7]).

Remark 2. The inequalities (31) and the second assumption in (32) hold if, e.g., $V_-^{(i)}$ is sufficiently large such that

$$\max\{1, U_-^{(i)}\} \leq \exp(-1)\, V_-^{(i)} \,. \tag{37}$$

More precisely, if (37) is satisfied, we can first choose the quotient $A_-^{(i)}/A_+^{(i)}$ such that

$$A_-^{(i)}/A_+^{(i)} \in [\max\{1, U_-^{(i)}\},\, \exp(-1)\, V_-^{(i)}] \,.$$

Then, without changing the quotient $A_-^{(i)}/A_+^{(i)}$, we choose $A_\pm^{(i)} > 0$ such that (31) holds. The conditions (32) and (37) are satisfied if the number $\mu^{(i)} > 0$ from (18) is sufficiently large which is the case if the length $L^{(i)}$ is not too long.

5 Proof of the Main Result Stated in Theorem 1

In this section we prove Theorem 1. The existence of a unique solution of (11) follows from Theorem 2.1 in [15] where initial-boundary value problems for first order quasilinear hyperbolic systems are studied (see also [5, 8]). For the proof of (36) we use the estimates (38), (40) and (41) for $\frac{d}{dt}\mathcal{E}^{(i)}(t)$, $\frac{d}{dt}\mathcal{D}^{(i)}(t)$ and $\frac{d}{dt}\mathcal{F}(t)$. The estimate (38) is the same as in [7] where $\alpha^{(i)} = \beta^{(i)} = 1/2$ ($i \in I$). The calculation of (40) is more complicated than in [7] where only constant delays are considered. Using integration by parts, Young's Inequality and the conditions (25) and (32), we obtain the following estimate for $\frac{d}{dt}\mathcal{E}^{(i)}(t)$ ($t \in [0, T]$):

$$\frac{d}{dt}\mathcal{E}^{(i)}(t) \leq -\alpha^{(i)}\beta^{(i)}\mu^{(i)}\mathcal{E}^{(i)}(t) + \Big[A_-^{(i)}h_-^{(i)}(x)(r_-^{(i)}(t,x))^2 - A_+^{(i)}h_+^{(i)}(x)(r_+^{(i)}(t,x))^2\Big]_{x=0}^{L^{(i)}} . \tag{38}$$

For a detailed calculation of (38) see [7]. For the derivative $\frac{d}{dt}\mathcal{D}^{(i)}(t)$ we get ($t \in [\overline{\tau}^{(i)}, T]$)

$$\begin{aligned}\tfrac{d}{dt}\mathcal{D}^{(i)}(t) = {} & 2\int_0^{\tau^{(i)}(t)} A_+^{(i)}\, h_+^{(i)}(L^{(i)})\, \exp(-\mu^{(i)}s)\, r_+^{(i)}(t-s, L^{(i)})\, \partial_t r_+^{(i)}(t-s, L^{(i)})\, ds \\ & + A_+^{(i)}\, h_+^{(i)}(L^{(i)})\, \exp(-\mu^{(i)}\tau^{(i)}(t))\, (r_+^{(i)}(t-\tau^{(i)}(t), L^{(i)}))^2\, \tfrac{d}{dt}\tau^{(i)}(t) \,.\end{aligned} \tag{39}$$

Using the equation $\partial_s r_+^{(i)}(t-s, L^{(i)}) = -\partial_t r_+^{(i)}(t-s, L^{(i)})$ and integration by parts, from (39) we obtain ($t \in [\overline{\tau}^{(i)}, T]$)

$$\begin{aligned}\tfrac{d}{dt}\mathcal{D}^{(i)}(t) = {} & -\mu^{(i)}\mathcal{D}^{(i)}(t) + A_+^{(i)}h_+^{(i)}(L^{(i)})(r_+^{(i)}(t, L^{(i)}))^2 \\ & - A_+^{(i)}h_+^{(i)}(L^{(i)})\exp(-\mu^{(i)}\tau^{(i)}(t))(r_+^{(i)}(t-\tau^{(i)}(t), L^{(i)}))^2(1 - \tfrac{d}{dt}\tau^{(i)}(t)) \,.\end{aligned}$$

Hence, the inequalities (14) and the definition of $\overline{\tau}^{(i)}$ and $\hat{\tau}^{(i)}$ in (15) imply ($t \in [\overline{\tau}^{(i)}, T]$)

$$\begin{aligned}\tfrac{d}{dt}\mathcal{D}^{(i)}(t) \leq {} & -\mu^{(i)}\mathcal{D}^{(i)}(t) + A_+^{(i)}h_+^{(i)}(L^{(i)})(r_+^{(i)}(t, L^{(i)}))^2 \\ & - A_+^{(i)}h_+^{(i)}(L^{(i)})\exp(-\mu^{(i)}\overline{\tau}^{(i)})(r_+^{(i)}(t-\tau^{(i)}(t), L^{(i)}))^2(1 - \hat{\tau}^{(i)}) \,.\end{aligned} \tag{40}$$

From (38) and (40) we obtain the following estimate for $\frac{d}{dt}\mathcal{F}(t)$ with $\eta = \min_{i\in I} \alpha^{(i)}\beta^{(i)}\mu^{(i)}$ $(t \in [\tau_{max},\, T])$:

$$\frac{d}{dt}\mathcal{F}(t) \;\leq\; -\,\eta\,\mathcal{F}(t) \,+\, B_0(t) \,+\, B_L(t) \tag{41}$$

with the boundary terms

$$\begin{aligned} B_0(t) &= \sum_{i\in I} A_+^{(i)} (r_+^{(i)}(t,0))^2 \;-\; A_-^{(i)} (r_-^{(i)}(t,0))^2 \;, \\ B_L(t) &= \sum_{i\in I} \Big[A_-^{(i)}\, h_-^{(i)}(L^{(i)})\, (r_-^{(i)}(t, L^{(i)}))^2 \\ &\quad -\; A_+^{(i)}\, h_+^{(i)}(L^{(i)})\; \exp(-\mu^{(i)}\overline{\tau}^{(i)})\, (1 - \hat{\tau}^{(i)})\, (r_+^{(i)}(t - \tau^{(i)}(t), L^{(i)}))^2 \Big] \;. \end{aligned}$$

The equation (13), the orthogonality of the matrix A_ω from (7) and the inequalities (31) yield that we have $B_0(t) \leq 0$ for all $t \in [\tau_{max},\, T]$. Furthermore, the boundary controls (16) and the inequality (33) guarantee $B_L(t) \leq 0$ for all $t \in [2\tau_{max},\, T]$. Thus, from (41) we obtain $(t \in [2\tau_{max},\, T])$

$$\frac{d}{dt}\mathcal{F}(t) \;\leq\; -\,\eta\,\mathcal{F}(t)$$

which implies the inequality (36).

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