Boundary Smoothness of the Solution of Maxwell's Equations of Electromagnetics

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Abstract We address the regularity of the solution to the time dependent Maxwell equations of electromagnetics in the case of metallic boundary condition under minimal regularity of the data. We extend the so-called extractor technique that we introduced in 1995 for wave equation in several cases (including the non-cylindrical case of moving domains for which the sharp-hidden regularity [10] was still an open problem). Concerning the electrical vector field we consider its normal component e at the boundary and, using a specific version of the so called pseudo-differential extractor (that we recently introduced in a different context), we obtain new sharp regularity results that are quantified in terms of curvature through the *oriented distance function* and all the intrinsic geometry we developed in the book [6].

1 Introduction

This paper deals with the regularity of the solution at the boundary of the 3D time-dependent solution E, H of Maxwell's equations of Electromagnetics. We show a *hidden regularity* result at the boundary for the electric field on a metallic obstacle. We consider a domain Ω with boundary Γ on which the boundary condition $E_\Gamma = 0$ is applied[1]. Assuming divergence-free initial data $E_i \in H^i(\Omega, \mathbb{R}^N)$, $i = 0, 1$, and divergence-free current $J \in L^2(0, \tau; L^2(\Omega, \mathbb{R}^N))$ we show that, at the boundary, the magnetic field verifies $H \in H^{1/2}(0, \tau; L^2(\Gamma, \mathbb{R}^3))$ while $\operatorname{curl} E \in$

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[1] For $N = 3$ this condition can be written $E \times n = 0$

$H^{-1/2}(0,\tau;L^2(\Gamma,\mathbb{R}^3))$. The proof makes use of the *Extractor technique* introduced at ICIAM 1995 [5] and in several papers ([1, 3, 2]); we first prove that $(DE.n)_\Gamma \in L^2(]0,\tau[\times\Gamma,\mathbb{R}^3)$, $E.n \in H^{1/2}(0,\tau;L^2(\Gamma))$ and $\nabla_\Gamma E.n \in H^{-1/2}(0,\tau;L^2(\Gamma))$. The proof of this last regularity follows a pseudo-differential extractor technique which is developed in a forthcoming paper [7].

2 Divergence-free Solutions of Maxwell and Wave Equations

As E is the electrical field, we deal with vector functions, say $E \in C^0([0,\tau],H^1(\Omega,\mathbb{R}^N))$, where Ω is a bounded smooth domain with boundary Γ and $I =]0,\tau[$ is the time interval. Throughout this paper we shall be concerned with divergence-free initial conditions E_0, E_1 and right-hand side F for the classical wave equation formulated in the cylindrical evolution domain $Q = I \times \Omega$. We shall discuss the boundary conditions on the lateral boundary $\Sigma = I \times \Gamma$.

2.1 Wave Deriving from Maxwell Equation

Assuming perfect media ($\varepsilon = \mu = 1$) the Ampère law is

$$\operatorname{curl}\mathbf{H} = \frac{\partial}{\partial t}E + J, \tag{1}$$

where J is the electric current density. The Faraday's law is

$$\operatorname{curl}E = -\frac{\partial}{\partial t}\mathbf{H}. \tag{2}$$

The conservation laws are

$$\operatorname{div}E = \rho, \quad \operatorname{div}\mathbf{H} = 0, \tag{3}$$

where ρ is the volume charge density. From (1) and (2), as $\operatorname{div}\operatorname{curl} = 0$, we obtain

$$\operatorname{div}J = -\operatorname{div}(E_t) = -\rho_t. \tag{4}$$

We assume that $\rho = 0$, which implies that $\operatorname{div}J = 0$. Under this assumption any E solving (1) is divergence-free as soon as the initial condition E_0 is. We shall also assume $\operatorname{div}E_0 = 0$ so that (6) will be a consequence of (1).

With $F = -J_t$, we similarly get $\operatorname{div}F = 0$ and E is solution of the usual Maxwell equation:

$$E_{tt} + \operatorname{curl}\operatorname{curl}E = F, \quad E(0) = E_0, \quad E_t(0) = E_1. \tag{5}$$

Lemma 1. *Assume that* $\operatorname{div}F = \operatorname{div}E_0 = \operatorname{div}E_1 = 0$*. Then any solution* E *to Maxwell equation (5) verifies the conservation condition (3) (with* $\rho = 0$*):*

$$\operatorname{div} E = 0. \tag{6}$$

We have the classical identity

$$\operatorname{curl}\operatorname{curl} E = -\Delta E + \nabla(\operatorname{div} E) \tag{7}$$

so that E is also solution of the following wave equation problem

$$E_{tt} - \Delta E = F, \quad E(0) = E_0, \quad E_t(0) = E_1. \tag{8}$$

2.2 Boundary Conditions

The physical boundary condition for metallic boundary is $E \times n = 0$ which can be written as the homogeneous Dirichlet condition on the tangential component of the field E:

$$E_\Gamma = 0 \text{ on } \Gamma. \tag{9}$$

We introduce the following Fourier-like boundary condition involving the mean curvature $\Delta b_\Omega = \lambda_1 + \lambda_2$ of the surface Γ

$$\Delta b_\Omega\, E.n + \langle DE.n, n\rangle = 0 \text{ on } \Gamma. \tag{10}$$

In flat pieces of the boundary this condition reduces to the usual Neumann condition.

Proposition 1. *Let E be a smooth element ($E \in \mathscr{H}^2$, see below) and the three divergence-free elements $(E_0, E_1, F) \in H^2(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathbb{R}^3) \times L^2(0, \tau; H^1(\Omega, R^3))$. Then we have the following conclusions.*

i) *Let E be solution to Maxwell-metallic system (5), (9). Then E solves the mixed wave problem (8), (9), (10) and, from Lemma 1, E solves also the free divergence condition (6).*
ii) *Let E be solution to the wave equation (8) with "metallic" b.c. (9). Then E verifies the Fourier-like condition (10) if and only if E verifies the free divergence condition (6).*
iii) *Let E be a divergence-free solution to the "metallic" wave problem (8), (6), (9), then E solves the Maxwell problem (5), (9), (10).*

Proof. We consider $e = \operatorname{div} E$; if E is solution to Maxwell problem (5) then e solves the scalar wave equation with initial conditions $e_i = \operatorname{div} E_i = 0,\ i = 0,1$ and right hand side $f = \operatorname{div} F = 0$. If E solves (10) then we get $e = 0$, as from the following result we get $e = 0$ on the boundary:

Lemma 2. *Let $E \in H^2(\Omega)$ solving the tangential Dirichlet condition (9), then we have the following expression for the trace of* $\operatorname{div} E$*:*

$$\operatorname{div} E|_\Gamma = \Delta b_\Omega \langle E, n\rangle + \langle DE.n, n\rangle \text{ on } \Gamma. \tag{11}$$

Proof. The divergence successively decomposes as follows at the boundary (see [13, 12]):

$$\begin{aligned} \operatorname{div} E|_\Gamma &= \operatorname{div}_\Gamma(E) + \langle DE.n, n\rangle = \operatorname{div}_\Gamma(E.n\,\mathbf{n}) + \operatorname{div}_\Gamma(E_\Gamma) + \langle DE.n, n\rangle \\ &= \langle \nabla_\Gamma(E.n), n\rangle + E.n \operatorname{div}_\Gamma(\mathbf{n}) + \operatorname{div}_\Gamma(E_\Gamma) + \langle DE.n, n\rangle . \end{aligned} \tag{12}$$

Obviously $\langle \nabla_\Gamma(E.n), n\rangle = 0$, the mean curvature of the surface Γ is $\Delta b_\Omega = \operatorname{div}_\Gamma(\mathbf{n})$ and if the field E satisfies the tangential Dirichlet condition (9) we get the following simple expression for the restriction to the boundary of the divergence:

$$\operatorname{div}(E)|_\Gamma = \Delta b_\Omega \langle E, n\rangle + \langle DE.n, n\rangle . \tag{13}$$

□

Then if E satisfies the extra "Fourier-like" condition (10) we get $e = 0$ on Γ, so that $e = 0$. □

2.3 The Wave-Maxwell Mixed Problem

From previous considerations, it follows that under the divergence-free assumption for the three data E_0, E_1, F, the following three problems are equivalent (in the sense that any smooth solution of one of them is solution to the two others): Maxwell problem (5), (9), Free-Wave problem (8), (6), (9), and Mixed-Wave problem (8), (9), (10). *We emphasize* that any solution to Maxwell problem satisfies the divergence-free condition (6) and the Fourier-like condition (10). Any solution to the Mixed-Wave problem satisfies (for free) the divergence-free condition (6). Any solution to the Free-Wave problem satisfies (for free) the Fourier-like condition (10). The object of this paper is to develop the proof of the following regularity result.

Theorem 1. *Let (E_0, E_1, J) be divergence-free vector fields in*

$$H^1(\Omega, \mathbb{R}^3)^2 \times L^2(\Omega, \mathbb{R}^3) \times H^1(I, L^2(\Omega, \mathbb{R}^3)) \tag{14}$$

with zero tangential components: $(E_0)_\Gamma = 0$. Assume also $\operatorname{curl} E_1 = 0$. The Maxwell problem (5), (9) has a unique solution

$$E \in C^0(\bar{I}, H^1(\Omega, \mathbb{R}^3)) \cap C^1(\bar{I}, L^2(\Omega, \mathbb{R}^3)) \tag{15}$$

verifying the boundary regularity:

$$\operatorname{curl} E|_\Gamma \in H^{-1/2}(I \times \Gamma, \mathbb{R}^3) \tag{16}$$

so that the magnetic field $\mathbf{H}$ *at the boundary verifies*

$$\mathbf{H}|_\Gamma \in H^{1/2}(I, L^2(\Gamma, R^3)). \tag{17}$$

Moreover, we have

$$E|_\Gamma \in H^{1/2}(I, L^2(\Gamma, \mathbb{R}^3)). \tag{18}$$

Furthermore, if $J|_\Gamma \in L^2(I, L^2(\Gamma))$, *from Ampère law (1) we obtain*

$$\operatorname{curl} \mathbf{H}|_\Gamma \in H^{-1/2}(I, L^2(\Gamma, \mathbb{R}^3)). \tag{19}$$

2.3.1 Tangential Decomposition

For any vector field $G \in H^1(\Omega, \mathbb{R}^N)$ denote by G_Γ the tangential part $G_\Gamma = G|_\Gamma - < G, n > \mathbf{n}$ and (see [12, 8, 6, 11]) consider its tangential Jacobian matrix $D_\Gamma G = D(G o p_\Gamma)|_\Gamma$ and its transpose D^*_Γ. To derive the regularity result we shall be concerned with the following three terms at the boundary:

$$(DE.n)_\Gamma,\ \nabla_\Gamma(E.n),\ E_t. \tag{20}$$

Lemma 3. *For all* $E \in H^2(\Omega, \mathbb{R}^N)$, *we have by direct computation:*

$$DE|_\Gamma = DE.n \otimes n + D_\Gamma E. \tag{21}$$

Obviously, as $E = E_\Gamma + \langle E, n \rangle\, n$, we have:

$$D_\Gamma E = D_\Gamma E_\Gamma + D_\Gamma(E.n\, n) \tag{22}$$

so that

$$E_\Gamma = 0 \quad \Rightarrow \quad D_\Gamma E = D_\Gamma(\langle E, n \rangle\, n). \tag{23}$$

Now as $D_\Gamma(\langle E, n \rangle\, n) = \langle E, n \rangle\, D_\Gamma(n) + n \otimes \nabla_\Gamma(\langle E, n \rangle)$ and as $D_\Gamma(n) = D^2 b_\Omega$, we get the following result.

Lemma 4. *Assume that* $E_\Gamma = 0$. *Then we have*

$$D_\Gamma E = \langle E, n \rangle\, D^2 b_\Omega|_\Gamma + n \otimes \nabla_\Gamma(\langle E, n \rangle). \tag{24}$$

Moreover as

$$\operatorname{div}_\Gamma E := \operatorname{div} E|_\Gamma - \langle DE.n, n \rangle \tag{25}$$

when $\operatorname{div} E = 0$, we get $\langle DE.n, n \rangle = -\operatorname{div}_\Gamma E$, and if, in addition, $E_\Gamma = 0$, we have $\langle DE.n, n \rangle = -\operatorname{div}_\Gamma(\langle E, n \rangle\, n)$, that is the following result.

Lemma 5. *Denote by* $H = \Delta b_\Omega$ *the mean curvature. Then*

$$E_\Gamma = 0 \text{ and } \operatorname{div} E = 0 \tag{26}$$

implies the following identities

i)

$$\langle DE.n, n \rangle = -H\, E.n, \tag{27}$$

ii)

$$DE.n = \langle DE.n,n\rangle\, n + (DE.n)_\Gamma = -H\, E.n\, n + (DE.n)_\Gamma \tag{28}$$

and

$$|DE.n|^2 = H^2|E.n|^2 + |(DE.n)_\Gamma|^2 \tag{29}$$

iii)

$$DE = -H\, E.n\, n\otimes n + \,(DE.n)_\Gamma \otimes n + E.n\, D^2 b + \, n\otimes \nabla_\Gamma(E.n), \tag{30}$$

iv)

$$DE..DE = H^2|E.n|^2 + |(DE.n)_\Gamma|^2 + |E.n|^2 D^2 b..D^2 b + |\nabla_\Gamma(E.n)|^2. \tag{31}$$

Proposition 2. *Let* $E \in H^2(\Omega,\mathbb{R}^N)$*,* $\operatorname{div} E = 0$*,* $E_\Gamma = 0$ *, then:*

$$DE..DE|_\Gamma = (H^2 + D^2 b..D^2 b)|E.n|^2 + |(DE.n)_\Gamma|^2 + |\nabla_\Gamma(E.n)|^2, \tag{32}$$

that is

$$DE..DE|_\Gamma = |DE.n|^2 + |E.n|^2 D^2 b..D^2 b + |\nabla_\Gamma(E.n)|^2. \tag{33}$$

2.4 Boundary Estimate of DE

Define $2\varepsilon \overset{\text{def}}{=} DE + D^*E$ and $2\sigma \overset{\text{def}}{=} DE - D^*E$ so that

$$DE = \varepsilon(E) + \sigma(E) \tag{34}$$

and

$$\|\operatorname{curl} E\|^2_{L^2(\Gamma,\mathbb{R}^3)} \le 4\,\|DE\|^2_{L^2(\Gamma,\mathbb{R}^{N^2})}. \tag{35}$$

From the decomposition (21) we have:

$$\|DE\|_{L^2(I,L^2(\Gamma,\mathbb{R}^3))} \le \|DE.n\otimes n\| + \|D^2 b\, E.n\|, \tag{36}$$

but

$$\|DE.n\otimes n\|^2 = \int_0^\tau \int_\Gamma (DE.n\otimes n)..(DE.n\otimes n)\, dt\, d\Gamma. \tag{37}$$

That is

$$\begin{aligned} \|DE.n\otimes n\|^2_{L^2(I,L^2(\Gamma,\mathbb{R}^3))} &\le \int_0^\tau \int_\Gamma |DE.n|^2\, dt\, d\Gamma \\ &= \int_0^\tau \int_\Gamma \{|(DE.n)_\Gamma|^2 + |< DE.n,n >|^2\}\, dt\, d\Gamma, \end{aligned} \tag{38}$$

but, as $\langle DE.n,n\rangle = -\langle E,n\rangle\, D^2 b_\Omega$, we get the estimate (35).

2.5 Extractor Identity

Let $I =]0, \tau[$ be the time interval and for any integer $k \geq 1$ define the spaces

$$H^k \stackrel{\text{def}}{=} C^0\left(\bar{I}, H^k(\Omega,\mathbb{R}^3)\right) \cap C^1\left(\bar{I}, H^{k-1}(\Omega,\mathbb{R}^3)\right), \tag{39}$$

$$\mathscr{H}^k \stackrel{\text{def}}{=} \left\{E \in H^k : \operatorname{div} E = 0,\ E_\Gamma = 0 \text{ on } \Gamma\right\}. \tag{40}$$

Let $F \in L^2(I, L^2(\Omega,\mathbb{R}^3))$, $E_0 \in H^1(\Omega,\mathbb{R}^3)$, $E_1 \in L^2(\Omega,\mathbb{R}^3)$ with $\operatorname{div} E_0 = \operatorname{div} E_1 = 0$. Consider $E \in \mathscr{H}^1$ solution of the equations

$$A.E := E_{tt} - \Delta E = F \in L^2(I, L^2(\Omega,\mathbb{R}^3)), \tag{41}$$

$$E(0) = E_0, \quad E_t(0) = E_1. \tag{42}$$

2.5.1 The Extractor $e(V)$

Let $E \in \mathscr{H}^2$, and $V \in C^0([0,\tau[, C^2(D,\mathbb{R}^N))$, $\langle V(t,.), n\rangle = 0$ on ∂D. Consider its flow mapping $T_s = T_s(V)$ and the derivative:

$$\mathbf{e}(V) \stackrel{\text{def}}{=} \frac{\partial}{\partial s}\mathscr{E}(V,s)\Big|_{s=0}, \tag{43}$$

where

$$\mathscr{E}(V,s) \stackrel{\text{def}}{=} \int_0^1 \int_{\Omega_s} \left(|E_t \circ T_s^{-1}|^2 - D(E \circ T_s^{-1})..D(E \circ T_s^{-1})\right) dx\, dt. \tag{44}$$

By change of variable

$$D(E \circ T_s^{-1}) \circ T_s = DE.DT_s^{-1} \tag{45}$$

we get the second expression

$$\mathscr{E}(V,s) = \int_0^1 \int_\Omega (|E_t|^2 - (DE.DT_s^{-1})..(DE.DT_s^{-1}))\, J(t)\, dx\, dt. \tag{46}$$

We have two expressions (44) and (46) for the same term $\mathscr{E}(V,s)$. The first one is an integral on a mobile domain $\Omega_s(V)$ while the second one is an integral over the fixed domain Ω. So taking the derivative with respect to the parameter s we shall obtain two different expressions for $\mathbf{e}$ that we shall respectively denote by $\mathbf{e_1}$ and $\mathbf{e_2}$.

2.5.2 Expression for $\mathbf{e_1}$

As the element E is smooth, $E \in \mathscr{H}^2$, we can directly apply the classical results from [12]. For simplicity, assume that $\operatorname{div} V = 0$ so that $J(t) = 1$. In this specific case we get

$$\mathbf{e} = \frac{\partial}{\partial s}\mathscr{E}\Big|_{s=0}, \tag{47}$$

and

$$\begin{aligned}\mathbf{e}_1 = 2\int_0^1\int_\Omega \{E_t.(-DE_t.V) - DE..D(-DE.V)\}\,dx\,dt \\ + \int_0^1\int_\Gamma \{|E_t|^2 - DE..DE\}v\,d\Gamma\,dt.\end{aligned} \tag{48}$$

2.5.3 Green-Stokes Theorem

Using integration by parts:

$$\begin{aligned}\int_0^1\int_\Omega \{DE..D(DE.V)\}\,dx\,dt = \int_0^1\int_\Omega \langle -\Delta E, DE.V\rangle\,dx\,dt \\ + \int_0^1\int_\Gamma \langle DE.n, DE.V\rangle\,d\Gamma(x)\,dt.\end{aligned} \tag{49}$$

2.5.4 Time Integration by Parts

Then

$$\begin{aligned}\int_0^1\int_\Omega E_t.(DE_t.V)\,dx\,dt = \int_0^1\int_\Omega (-E_{tt}.(DE.V) + E_t.(DE.W))\,dx\,dt \\ - \int_\Omega E_t(0).(DE(0).W)\,dx.\end{aligned} \tag{50}$$

Furthermore, assuming that the initial condition is of the form

$$E_0 \in H^1(\Omega, \mathbb{R}^3), \quad E_1 \in L^2(\Omega, \mathbb{R}^3), \tag{51}$$

we get

$$\begin{aligned}\mathbf{e}_1 = & 2\int_0^1\int_\Omega \{E_{tt}.(DE.V) - E_t.(DE.W) - \langle \Delta E, DE.V\rangle\}\,dx\,dt \\ & + 2\int_\Omega E_1.(DE_0.W)\,dx \\ & + \int_0^1\int_\Gamma \{(|E_t|^2 - DE..DE)\,\langle V, n\rangle + 2\,\langle DE.n, DE.V\rangle\}\,d\Gamma(x)\,dt.\end{aligned} \tag{52}$$

The discussion is now on the last boundary integral.

2.5.5 Specific Choice for V at the Boundary

As the boundary $\Gamma = \partial\Omega \in C^2$ we can apply all intrinsic geometry material introduced in [6]. Denoting by $p = p_\Gamma$ the projection mapping onto the manifold Γ (which is smoothly defined in a tubular neighborhood of Γ) we consider the oriented distance function $b = b_\Omega = d_{\Omega^c} - d_\Omega$ where $\Omega^c = \mathbb{R}^N \setminus \Omega$, and its "localized version" defined as follows (see [4]): let $h > 0$ be "a small" positive number and $\rho_h(.) \geq 0$ be a cutting scalar smooth function such that $\rho_h(z) = 0$ when $|z| > h$ and $\rho(z) = 1$ when $|z| < h/2$. Then set

$$b^h_\Omega \stackrel{\text{def}}{=} \rho_h \circ b_\Omega \tag{53}$$

and define the associate localized projection mapping

$$p_h \stackrel{\text{def}}{=} I_d - b^h_\Omega \nabla b^h_\Omega \tag{54}$$

smoothly defined in the tubular neighborhood

$$\mathscr{U}_h(\Gamma) \stackrel{\text{def}}{=} \{x \in D : |b_\Omega(x)| < h\}. \tag{55}$$

Let any smooth element $v \in C^0(\Gamma)$ be given and consider the vector field V of the following form

$$V(t,x) \stackrel{\text{def}}{=} W(x)(1-t), \quad W(x) \stackrel{\text{def}}{=} v \circ p_h \nabla b^h_\Omega. \tag{56}$$

Then the last term (boundary integral) in (52) takes the following form:

$$\int_0^1 \int_\Gamma \{(|E_t|^2 - DE..DE) + 2\langle DE.n, DE.n\rangle\} v(1-t)\, d\Gamma(x)\, dt. \tag{57}$$

We get:

$$\begin{aligned}\mathbf{e}_1 = &\int_0^1 \int_\Gamma (|E_t|^2 - DE..DE + 2|DE.n|^2)\, v\, d\Gamma\, dt \\ &+ 2\int_Q (E_{tt}.DE.V - \langle \Delta E, DE.V\rangle\, dx\, dt - \int_\Omega \langle E_t(0), DE(0).W\rangle\, dx.\end{aligned} \tag{58}$$

As from (33) we have

$$DE..DE = |DE.n|^2 + D^2 b_\Omega..D^2 b_\Omega |E.n|^2 + |\nabla_\Gamma E.n|^2 \tag{59}$$

and as

$$|DE.n|^2 = |(DE.n)_\Gamma|^2 + (\Delta b_\Omega)^2 |E.n|^2 \tag{60}$$

we obtain the following result.

Proposition 3.

$$\begin{aligned}\mathbf{e}_1 = \int_0^1 \int_\Gamma (\tau - t)\{|E_t|^2 + |(DE.n)_\Gamma|^2 - |\nabla_\Gamma (E.n)|^2 \\ + |E.n|^2 (H^2 - D^2 b..D^2 b)\} v\, d\Gamma\, dt \\ + 2\int_Q \langle A.E, DE.V\rangle\, dQ - 2\int_\Omega \langle E_1, D(E_0).W\rangle\, dx.\end{aligned} \tag{61}$$

2.5.6 Second Expression for e

From (46) we obtain the s derivative as a distributed integral term as follows

$$\mathbf{e}_2 = \int_Q \left\{(|E_t|^2 - DE..DE)\operatorname{div} V(0) - 2DE..(-DE.DV)\right\}\, dx\, dt. \tag{62}$$

2.5.7 Extractor Identity

As $\mathbf{e} = \mathbf{e}_1 = \mathbf{e}_2$ we get

$$\begin{aligned}&\int_\Sigma (\tau - t)\{(|E_t|^2 - |\nabla_\Gamma (E.n)|^2 + |(DE.n)_\Gamma|^2 + |E.n|^2 (H^2 - D^2 b..D^2 b))\} v\, d\Sigma \\ &= \int_Q \{(|E_t|^2 - DE..DE)\operatorname{div} V - 2DE..(-DE.DV)\}\, dx\, dt \\ &\quad - \int_Q 2(E_{tt} - \Delta E).DE.V\, dQ + \int_\Omega 2\langle E_1, DE_0.W\rangle\, dx.\end{aligned} \tag{63}$$

That is

$$\begin{aligned}&\int_\Sigma (\tau - t)\{(|E_t|^2 - |\nabla_\Gamma (E.n)|^2 + (DE.n)_\Gamma|^2)\} v\, d\Sigma \\ &= \int_Q \{(|E_t|^2 - DE..DE)\operatorname{div} V - 2DE..(-DE.DV)\}\, dx\, dt \\ &\quad - 2\int_Q 2\langle A.E, DE.V\rangle\, dQ + \int_\Omega 2\langle E_1, DE_0.W\rangle\, dx \\ &\quad + \int_\Sigma (1 - t)|E.n|^2 (D^2 b..D^2 b - H^2)\, v\, d\Sigma.\end{aligned} \tag{64}$$

Notice that the curvature terms

$$D^2 b..D^2 b - H^2 = \lambda_1^2 + \lambda_2^2 - (\lambda_1 + \lambda_2)^2 = -2\kappa, \tag{65}$$

where $\kappa = \lambda_1 \lambda_2$ is the *Gauss curvature* of the boundary Γ.

3 Regularity at the Boundary

We apply twice this last identity.

3.1 Tangential Field E^τ

In a first step consider the "tangential vector field" obtained as $E^\tau \overset{\text{def}}{=} E - E.\nabla b^h_\Omega \, \nabla b^h_\Omega$. We get

$$E^\tau_{tt} - \Delta E^\tau = (E_{tt} - \Delta E) - (E_{tt} - \Delta E).\nabla b^h_\Omega \, \nabla b^h_\Omega + C. \tag{66}$$

That is $A.E^\tau = (A.E)^\tau + C$, where the commutator $C \in L^2(0,T,L^2(\Omega,\mathbb{R}^3))$ is given by

$$C \overset{\text{def}}{=} -E.\Delta b^h_\Omega \, \nabla b^h_\Omega - 2D^2 b^h_\Omega.\nabla b^h_\Omega \, \nabla b^h_\Omega - E.\nabla b^h_\Omega d^2 b^h_\Omega - 2D^2 b^h_\Omega.\nabla(E.b^h_\Omega). \tag{67}$$

The conclusion formally derives as follows: as $E^\tau \in L^2(I,H^1(\Omega,\mathbb{R}^3))$ we get the traces terms

$$E^\tau.n = E^\tau_t = 0 \in L^2(I,H^{1/2}(\Gamma)). \tag{68}$$

Since $e_1 = e_2$, we conclude by choosing the vector field of the form

$$V(t,x) = (\tau - t)\,\nabla b^h_\Omega = (\tau - t)\,\rho'_h \circ b_\Omega \, \nabla b_\Omega. \tag{69}$$

That is $v = 1$ and as for $0 < t \le \tau/2$ we have $\tau/2 \le \tau - t$, we get:

$$\begin{aligned}
&\tau/2 \int_0^{\tau/2} \int_\Gamma |(DE^\tau.n)_\Gamma|^2 \, d\Gamma \, dt \le \int_0^{\tau/2} (\tau - t) \int_\Gamma |(DE^\tau.n)_\Gamma|^2 \, d\Gamma \, dt \\
&\le \int_0^{\tau} (\tau - t) \int_\Gamma |(DE^\tau.n)_\Gamma|^2 \, d\Gamma \, dt \\
&= \int_Q (\tau - t) \left\{ (|E^\tau_t|^2 - DE^\tau..DE^\tau)\,\mathrm{div}\,(\nabla b^h_\Omega) - 2DE^\tau..(-DE^\tau.D(\nabla b^h_\Omega)) \right\} dQ \\
&\quad - 2\int_Q (\tau - t) \left\langle A.E^\tau, DE^\tau.(\nabla b^h_\Omega) \right\rangle dQ + \int_\Omega 2 \left\langle E^\tau_1, DE^\tau_0.(\nabla b^h_\Omega) \right\rangle dx.
\end{aligned} \tag{70}$$

As for $0 < t < \tau$ we have $2/\tau(\tau - t) \le 2$ we get, with $T = \tau/2$

$$\begin{aligned}
&\int_0^T \int_\Gamma |(DE^\tau.n)_\Gamma|^2 \, d\Gamma \, dt \\
&\le 2\int_0^{2T} \int_\Omega \left\{ (|E^\tau_t|^2 - DE^\tau..DE^\tau)\,\mathrm{div}\,(\nabla b^h_\Omega) - 2DE^\tau..(-DE^\tau.D(\nabla b^h_\Omega)) \right\} dx\,dt \\
&\quad - 4\int_0^{2T} \int_\Omega \left\langle A.E^\tau, DE^\tau.(\nabla b^h_\Omega) \right\rangle dx\,dt + 4/T \int_\Omega \left\langle E^\tau_1, DE^\tau_0.(\nabla b^h_\Omega) \right\rangle dx,
\end{aligned} \tag{71}$$

there exists a constant $M > 0$ such that

$$\int_0^T \int_\Gamma \{|E_t^\tau|^2 + |(DE^\tau.n)_\Gamma|^2\} \, d\Gamma \, dt$$
$$\le M \, \|\nabla b_\Omega^h\|_{W^{1,\infty}(\Omega,\mathbb{R}^N)} \cdots$$
$$\cdot \left\{ \|E^\tau\|^2_{\mathscr{H}^1(0,2T)} + \, |A.E^\tau|_{L^2([0,2T]\times\Omega,\mathbb{R}^3)} + 1/T\left(|E_0^\tau|^2_{H^1(\Omega,\mathbb{R}^3)} + |E_1^\tau|^2_{L^2(\Omega,\mathbb{R}^3)}\right) \right\}. \tag{72}$$

Notice that

$$\nabla b_\Omega^h = \rho_h' \circ b_\Omega \nabla b_\Omega, \tag{73}$$

so that

$$\|\nabla b_\Omega^h\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^N)} \le \mathrm{Max}_{0\le s\le h} |\rho_h'(s)|. \tag{74}$$

Moreover

$$D^2 b_\Omega^h = D(\rho_h' \circ b_\Omega \nabla b_\Omega) = \rho_h'' \circ b_\Omega \nabla b_\Omega \times \nabla b_\Omega + \rho_h' \circ b_\Omega D^2 b_\Omega \tag{75}$$

so that

$$\|D^2 b_\Omega^h\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^{N^2})} \le \mathrm{Max}_{0\le s\le h}|\rho_h''(s)| + \mathrm{Max}_{0\le s\le h}|\rho_h'(s)| \; \|D^2 b_\Omega\|_{L^\infty(\mathscr{U}_h(\Gamma),\mathbb{R}^{N^2})}. \tag{76}$$

By choice of ρ_h in the form $\rho_h(s) = f(2s/h - 1)$ when $h/2 < s < h$ and $F(x) = 2x^3 - 3x^2 + 1$, we obtain

$$\|\rho_h\|_{C^2([0,h])} \le \frac{8}{h^2}. \tag{77}$$

So the previous estimate is in the form

$$\|D^2 b_\Omega^h\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^{N^2})} \le C_0 \frac{1}{h^2} \, \|D^2 b_\Omega\|_{L^\infty(\mathscr{U}_h(\Gamma),\mathbb{R}^{N^2})} \tag{78}$$

for the *larger h* such that the following condition holds

$$D^2 b_\Omega \in L^\infty(\mathscr{U}_h(\Gamma), \mathbb{R}^{N^2}). \tag{79}$$

3.1.1 Regularity Result for E^τ

Proposition 4. *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ being a C^2 manifold. Let h verify condition (79).*

There exists a constant $M > 0$ such that for any data $(E_0, E_1, F) \in L^2(\Omega,\mathbb{R}^3) \times H^1(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathbb{R}^3)$, the vector

$$E^\tau \in \mathscr{H}^1(0,2\tau) \stackrel{\text{def}}{=} C^0\left([0,2\tau], H^1(\Omega,\mathbb{R}^3)\right) \cap C^1\left([0,2\tau], L^2(\Omega,\mathbb{R}^3)\right) \tag{80}$$

verifies

$$(DE^{\tau}.n)_{\Gamma} \in L^2(0,\tau;L^2(\Gamma,\mathbb{R}^3)) \tag{81}$$

and

$$\begin{aligned}&\int_0^T \int_{\Gamma} \{|(DE^{\tau}.n)_{\Gamma}|^2\} d\Gamma\, dt \\ \leq& M\, \|\nabla b_{\Omega}^h\|_{W^{1,\infty}(\Omega,\mathbb{R}^N)} \cdots \\ &\cdot \{\|E\|^2_{\mathscr{H}^1(0,2T)} + |F|_{L^2([0,2T]\times\Omega,\mathbb{R}^3)} + 1/T\, |E_0^{\tau}|^2_{H^1(\Omega,\mathbb{R}^3)} + 1/T\, |E_1^{\tau}|^2_{L^2(\Omega,\mathbb{R}^3)}\}\end{aligned} \tag{82}$$

It can be verified that

$$D(E^{\tau}).n = (DE.n)_{\Gamma}. \tag{83}$$

3.2 The Normal Vector Field e

Set

$$e \overset{\text{def}}{=} E.\nabla b_{\Omega}^h. \tag{84}$$

Lemma 6.

$$e_{tt} - \Delta e = (E_{tt} - \Delta E).\nabla b_{\Omega}^h + \theta, \tag{85}$$

where

$$\theta = D^2 b_{\Omega}^h..DE + \operatorname{div}(D^2 b_{\Omega}^h.E) \quad \text{and} \quad \frac{\partial}{\partial n} e = \langle DE.n, n\rangle = -\Delta b_{\Omega}\, e \text{ on } \Gamma. \tag{86}$$

Then e is solution of the wave problem:

$$e_{tt} - \Delta e = \Theta, \tag{87}$$

where

$$\Theta = F.\nabla b_{\Omega}^h + D^2 b_{\Omega}^h..DE + \operatorname{div}(D^2 b_{\Omega}^h.E). \tag{88}$$

3.3 Extension to $\mathbb{R}$

Let

$$\rho \in C^2(\mathbb{R}), \quad \rho \geq 0, \quad \operatorname{supp} \rho \subset [-2\tau, +2\tau], \quad \rho = 1 \text{ on } [-\tau, +\tau]. \tag{89}$$

Define

$$\tilde{e} \overset{\text{def}}{=} \rho(t)e(t), \quad t \geq 0, \quad \tilde{e} = \rho(t)(e_0 + te_1), \quad t < 0, \tag{90}$$

which turns to be solution on $\mathbb{R}$ to the wave problem

$$\tilde{e}_{tt} - \Delta\tilde{e} = H \quad \text{and} \quad \frac{\partial}{\partial n}\tilde{e} = g, \tag{91}$$

where

$$g \stackrel{\text{def}}{=} \begin{cases} -\Delta b_\Omega \tilde{e} & \text{for } t > 0 \\ \rho(t)(\frac{\partial}{\partial n}e_0 + t\frac{\partial}{\partial n}e_1) & \text{for } t < 0, \end{cases} \quad \text{on } \Gamma \tag{92}$$

and $H \in L^2(\mathbb{R}, L^2(\Omega))$ verifies

$$H \stackrel{\text{def}}{=} \begin{cases} \rho(t)\Theta + \rho'' e + 2\rho' \frac{\partial}{\partial t} e & \text{for } t > 0 \\ \rho''(e_0 + te_1) + 2\rho' e_1 - \rho(\Delta e_0 + t\Delta e_1) & \text{for } t < 0. \end{cases} \tag{93}$$

4 Fourier Transform

Define

$$z(\zeta)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} exp(-i\zeta t)\,\tilde{e}(t,x)\,dt, \tag{94}$$

which turns to be solution to

$$\frac{\partial}{\partial n} z = \mathscr{F}.g \text{ on } \Gamma. \tag{95}$$

Consider the perturbed domain $\Omega_s = T_s(V)(\Omega)$ with boundary $\Gamma_s = T_s(V)(\Gamma)$, the following integral and derivative

$$\mathscr{E}(s,V) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\zeta \int_{\Omega_s(V)} \left(|\zeta||z \circ T_s(V)^{-1}|^2 + \frac{1}{1+|\zeta|}|\nabla(z \circ T_s(V)^{-1})|^2 \right) dx, \tag{96}$$

$$e \stackrel{\text{def}}{=} \frac{d}{ds}\mathscr{E}(s,V)\bigg|_{s=0}, \tag{97}$$

and compute the derivative in the two different ways.

4.0.1 By Moving Domain Derivative

Let

$$e_1 \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left(|\zeta| 2\mathscr{R}e\{\langle z, \nabla\bar{z}.(-V)\rangle\} + \frac{1}{1+|\zeta|} 2\mathscr{R}e\{\langle \nabla z, \nabla(\nabla\bar{z}(-V))\rangle\} \right) dx$$
$$+ \int_{-\infty}^{+\infty} d\zeta \left(\int_{\Gamma} \left\{ |\zeta|\,|z|^2 + \frac{1}{1+|\zeta|}\,|\nabla z|^2 \right\} \langle V, n\rangle \, d\Gamma(x) \right). \tag{98}$$

By Stokes theorem we get,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} 2\mathscr{R}e\{\nabla(z).\nabla(\nabla\bar{z}(-V))\}\, dx \\
&\quad = \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} 2\mathscr{R}e\{\Delta(z), (\nabla\bar{z}.V)\}\, dx \\
&\quad - \int_{-\infty}^{+\infty} \int_{\Gamma} \frac{1}{1+|\zeta|} 2\mathscr{R}e\{\langle \nabla z.n, \nabla\bar{z}.V\rangle\}\, d\Gamma\, dt.
\end{aligned} \tag{99}$$

As $V = vn$ on Γ, we get for the last term:

$$-\int_{-\infty}^{+\infty} \int_{\Gamma} \frac{1}{1+|\zeta|} 2\mathscr{R}e\{\langle \nabla z.n, \nabla\bar{z}.n\rangle\}\, v\, d\Gamma\, dt, \tag{100}$$

but on Γ we have

$$\langle \nabla z.n, \nabla\bar{z}.n\rangle = |\mathscr{F}.g|^2. \tag{101}$$

Finally, we get

$$e_1 = \int_{-\infty}^{+\infty} \int_{\Gamma} \{|\zeta|\,|z|^2 + \frac{1}{1+|\zeta|} \mathscr{R}e\{\langle \nabla z, \nabla\bar{z}\rangle\} - 2|\mathscr{F}.g|^2\}\, v\, d\Gamma\, dt$$
$$+ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} (|\zeta| 2\mathscr{R}e\{\langle z, \nabla\bar{z}.(-V)\rangle\} + \frac{1}{1+|\zeta|} 2\mathscr{R}e\{\Delta z, (\nabla\bar{z}.V)\})\, dx. \tag{102}$$

Then

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \int_{\Gamma} \{|\zeta|\,|z|^2 + \frac{1}{1+|\zeta|}\,|\nabla_{\Gamma} z|^2\} v \\
&\quad = \int_{-\infty}^{+\infty} \int_{\Gamma} \frac{1}{1+|\zeta|}\,|\mathscr{F}.g|^2\, d\Gamma dt \\
&\quad - \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} 2\mathscr{R}e\{|\zeta|^2 z + \mathscr{F}.H(\nabla\bar{z}.V)\}\, dx \\
&\quad + \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} |\zeta|\, 2\mathscr{R}e\{\langle z, \nabla\bar{z}.V\rangle\}\, dx + e_2.
\end{aligned} \tag{103}$$

Hence there exists $M > 0$ such that

$$\int_{-\infty}^{+\infty} \int_{\Gamma} \{|\zeta|\,|z|^2 + \frac{1}{1+|\zeta|}\,|\nabla_{\Gamma} z|^2\} v \le M \left\{ \|z\|^2_{L^2(\mathbb{R},H^1(\Omega))} + \|z\|^2_{L^2(\mathbb{R},L^2(\Gamma))} \right\}. \tag{104}$$

We have $\sqrt{|\zeta|}z \in L^2(\mathbb{R}_\zeta, L^2(\Gamma))$ and $\frac{1}{\sqrt{|\zeta|}}\nabla_\Gamma z \in L^2(\mathbb{R}_\zeta, L^2(\Gamma, \mathbb{R}^N))$. By a density argument, we conclude that

$$E.n \in H^{1/2}(I, L^2(\Gamma)) \cap L^2(I, H^{1/2}(\Gamma)) \tag{105}$$

$$\nabla_\Gamma(E.n) \in H^{-1/2}(I, L^2(\Gamma, \mathbb{R}^N)). \tag{106}$$

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