On A Way to Improve Cyber-Insurer Profits *When a Security Vendor Becomes the Cyber-Insurer*

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Abstract*—Current research on cyber-insurance has mainly been about studying the market success of an insurance-driven security ecosystem. Such an ecosystem comprises of a set of market elements (e.g., cyber-insurers, network users, security vendors (SVs), regulatory agencies, etc.,) that coexist together as a system with the success goal of mutually satisfying each other's interests. However, existing works have not explicitly considered SVs as market elements, and have analytically proved the moderate/no success[1] of cyber-insurance markets due to insurers not satisfying their interests in making zero expected profits at times. In this paper, we model a security vendor (e.g., *Symantec, Microsoft*) as a cyber-insurer, thereby making the former as an explicit market element. We then propose a novel consumer pricing mechanism for SVs based on their client/consumer logical network and consumer security investment amounts, with the goal to improve SV profits. Our simulation results show that SVs could improve up to 25% of their current profit margins by accounting for client location and investment information. A fraction of the extra profits could then be used up by the SVs to recover costs related to providing insurance coverage to their clients, and also to always make strictly positive profits as insurers. Finally, we propose a discrete-time sequential dynamic pricing strategy for an SV, and show that it leads to improved SV profits when compared to a static pricing strategy. Our work demonstrates one particular way for a cyber-insurance market to be successful on a more than moderate scale.

***Keywords:* insurance, security vendor, pricing, centrality**

I. Introduction

The infrastructure, the users, and the services offered on computer networks today are all subject to a wide variety of risks. These risks include distributed denial of service attacks, intrusions of various kinds, eavesdropping, hacking, phishing, worms, viruses, spams, etc. In order to counter the threats posed by the risks, network users[2] have traditionally resorted to antivirus and anti-spam softwares, firewalls, intrusion-detection systems (IDSs), and other add-ons to reduce the likelihood of being affected by threats. In practice, a large industry (companies like *Symantec, McAfee,* etc.) as well as considerable research efforts are currently centered around developing and deploying tools and techniques to detect threats and anomalies in order to protect the cyber infrastructure and its users from the negative impact of the anomalies.

Inspite of improvements in risk protection techniques over the last decade due to hardware, software and cryptographic methodologies, it is impossible to achieve a perfect/near-perfect cyber-security protection [2][7]. The impossibility arises due to a number of reasons: (i) scarce existence of sound technical solutions, (ii) difficulty in designing solutions catered to varied intentions behind network attacks, (iii) misaligned incentives between network users, security product vendors, and regulatory authorities regarding each taking appropriate liabilities to protect the network, (iv) network users taking advantage of the positive security effects generated by other user investments in security, in turn themselves not investing in security and resulting in the free-riding problem, (v) customer lock-in and first mover effects of vulnerable security products, (vi) difficulty to measure risks resulting in challenges to designing pertinent risk removal solutions, (vii) the problem of a lemons market [1], whereby security vendors have no incentive to release robust products in the market, and (viii) liability shell games played by product vendors. In view of the above mentioned inevitable barriers to 100% risk mitigation, the need arises for alternative methods of risk management in cyberspace. In this regard, some security researchers in the recent past have identified *cyber-insurance* as a potential tool for effective risk management.

Cyber-insurance is a risk management technique via which network user risks are transferred to an insurance company (e.g., ISP, cloud provider.), in return for a fee, i.e., the *insurance premium*. Proponents of cyber-insurance believe that cyber-insurance would lead to the design of insurance contracts that would shift appropriate amounts of self-defense liability on the clients, thereby making the cyberspace more robust. Here the term 'self-defense' implies the efforts by a network user to secure their system through technical solutions such as anti-virus and anti-spam softwares, firewalls, using secure operating systems, etc. Cyber-insurance has also the potential to be a market solution that can align with economic incentives of cyber-insurers, users (individuals/organizations), policy makers, and security software vendors, i.e., the cyber-insurers will earn profit from appropriately pricing premiums, network users will seek to hedge potential losses by jointly buying insurance and investing in self-defense mechanisms, the policy makers would ensure the increase in overall network security, and the security software vendors could go ahead with their first-mover and lock-in strategies as well as experience an increase in their product sales via forming alliances with cyber-insurers.

A. Research Motivation

Recent research works on cyber-insurance [6][7][10] have mathematically shown the existence of inefficient insurance markets. Intuitively, an efficient market (see Section I-C.) is one where all stakeholders (market elements) mutually satisfy their interests. These works state that cyber-insurance makes every stakeholder (see Section I-C.) satisfied apart from the regulatory agency (e.g., government) and sometimes the cyber-insurer itself. The regulatory agency is unsatisfied as overall network robustness is sub-optimal due to network users not optimally investing in self-defense mechanisms, whereas a cyber-insurer is unsatisfied due to it potentially making zero expected profits at times. Lelarge et.al. in [7] recommended the use of fines and rebates on cyber-insurance contracts to make

[1] Currently, the US cyber-insurance market is worth $800 million with only certain industries and organizations buying insurance to cover for losses due to cyber-threats. A more than moderate success would imply a cyber-insurance market where common people, apart from industries and organizations, would also buy insurance.

[2] The term 'users' may refer to both, individuals and organizations.

each user invest optimally in self-defense investments and make the network optimally robust. However, there is still no work that guarantees the strict positiveness of insurer profits at all times. The notion of making zero expected profits at times is enough for cyber-insurers to opt out of the market in future, leading to an insurance market failure. The important question that thus arises is *is there a way for a cyber-insurer to make strictly positive profits at all times, and at the same time ensure optimal network robustness?* A positive answer to this question would imply cyber-insurance market success on a larger than moderate scale.

In addition, in a correlated risk environment such as the Internet, an insurers cannot afford to be risk-neutral as there are chances it might go bankrupt due to expected aggregate losses in a period being more than what it could afford to compensate. As a result it might hold a safety capital for a certain cost in order to prevent itself from going bankrupt [3]. The question that arises here is *how can the cyber-insurers recover costs of buying safety capital?*

The above two questions motivate us to investigate a way in which cyber-insurers can always make profits and recover their costs to provide insurance coverage to clients, and at the same time ensure optimal network robustness.

B. Research Contributions

We make the following research contributions in this paper.

- We model risk-averse security vendors as cyber-insurers and propose a one-period static product pricing scheme for their consumers based on the consumers' logical network and their security investment amounts. Our proposed approach (i) potentially increases the current profit margins of SVs upto 25% and allows an SV to make strictly positive profits at all times, solely as an insurer, (ii) ensures the state of optimal network robustness, and (iii) allows risk-averse SV insurers to recover costs such as ones related to buying safety capital. (See Section III.)
- We extend the one-period static pricing scheme above to a sequential multi-stage dynamic pricing scheme and show that it results in an SV cyber-insurer making more profits compared to that in the static one-period pricing scheme. (See Section IV.)

C. Basic Economic Concepts

In this section we briefly describe some basic economic concepts relevant to the paper. Additional details can be found in a standard economics text such as [8].

externality: An externality is an effect (positive or negative) of a purchase of self-defense investments by a set of users (individuals or organizations) on other users whose interests were not taken into account while making the investments. In this work, the effects are improvements in individual security of network users who are connected to the users investing in self-defense.

risk probability: It is the probability of a network user being successfully attacked by a cyber-threat and incurring a loss of a particular amount.

market: It is a regulated platform where cyber-insurance products are traded with insurance clients, i.e., the network users. A market may be perfectly competitive, oligopolistic, or monopolistic. In a perfectly competitive market there exists a large number of buyers (those insured) and sellers (insurers) that are small relative to the size of the overall market. The exact number of buyers and sellers required for a competitive market is not specified, but a competitive market has enough buyers and sellers that no one buyer or seller can exert any significant influence on premium pricing in the market. On the contrary, in a monopolistic market, the single insurer has the power to set client premiums to its liking. An oligopolistic insurance market is a special type of a competitive market where multiple insurance firms exist in a manner so that each insurer can set client premiums to its liking.

Symbol	Meaning
$u_i(\cdot)$	utility of user (consumer) i in consumer-seller model
N	number of consumers of an SV
h_{ij}	externality effect of user j on user i
x_i	amount of self-defense good consumed by user i
G	matrix representing externality values between user pairs
$\overrightarrow{x_{-i}}$	vector of self-defense amounts of users apart from i
$B(\cdot)$	Bonacich centrality vector of users in a logical network
c	constant marginal manufacturing cost to SV
p_i	price per unit of self-defense good consumed by i
x_i^k	self-defense good consumed by user i in round k
p_i^k	price per unit of self-defense to user i in round k
$\overrightarrow{x_i^{1:k-1}}$	vector of user i's self-defense amounts up to round $k-1$
$\overrightarrow{p_i^{1:k-1}}$	vector of user i's prices up to round $k-1$
ϕ^k	sequential order in which SV visit its consumers in round k

TABLE I
LIST OF IMPORTANT SYMBOLS

stakeholders: The stakeholders in a cyber-insurance market refer to entities whose interests are affected by the dynamics of market operation. In our work we assume that the entities are the network users, a regulatory agency such as the government, and security vendors (also the cyber-insurers) such as Symantec and Microsoft. When all stakeholders in a market are satisfied, it results in a market success.

market efficiency: A cyber-insurance market is called efficient if the social welfare of all insured network users is maximized at the market equilibrium. The market is inefficient if it fails to achieve this condition. Here 'social welfare' refers to the sum of the net utilities of insured network users after investing in self-defense and/or cyber-insurance. At the maximum social welfare state, the moral hazard problem[3] in cyber-insurance is alleviated, i.e., network users adopt safe Internet browsing habits even after after getting insured, knowing that they would be covered by their insurers.

II. System Setup and Model

In this section we propose our system model. First, we qualitatively describe the insurance environment under which our proposed SV pricing mechanisms could operate. We then follow it up with a description of the SV pricing environment. Finally, we define our system model. A list of important symbols relevant to the paper is shown in Table I.

A. Insurance Setting (Environment)

We consider a system of security vendors existing in a market and offering cyber-insurance solutions to their clients. Each client is locked[4] with his corresponding SV with respect to using security products manufactured by the SV, i.e., he does not use products manufactured by any other firm for self-defense purposes. A consumer (network user) of security products may or may not buy cyber-insurance, i.e, buying cyber-insurance is not made mandatory. In general, a risk-averse consumer would want to buy insurance, whereas risk-loving or risk-neutral users would not care that much about buying insurance. A consumer buying cyber-insurance is provided a full coverage by his SV on facing a loss and is charged a premium which is not necessarily fair, i.e., the expected value of the loss. Each SV also premium discriminates its clients in the form of charging

[3]Moral hazard is a well known problem in insurance literature where an insured could behave recklessly after getting insured, knowing that he could get coverage from his insurance company for the losses faced.

[4]A client may lock-in with a security vendor due to various aspects such as reputation, service quality etc.

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fines/rebates atop premiums. Here we assume that the SVs are located in a competitive market setting where each SV can afford to premium discriminate its clients in a certain manner without the fear of losing consumer demand to other SVs. Based on works in [6][7], premium discriminating cyber-insurance clients is one way to alleviate the moral hazard problem, enables the insured's to optimally invest in self-defense investments, and maximizes the social welfare of the insured's in the network. We assume that each cyber-insurer in the market wants to maximize social welfare of its clients as a regulatory constraint imposed upon it by a regulator such as the government, and therefore premium discriminates its clients.

Remark: Since our focus in the paper is on SV pricing, and the pricing step is a pre-cursor to the client premium charging step, we do not mathematically model the insurance aspects in the paper. By the term 'pre-cursor' we mean that the net premiums (including fines and rebates) charged to clients will depend on the self-defense investment amounts of users which in turn depend on the prices set by the SV on its security products. However, it is important to state down the type of insurance environment/s where our proposed pricing mechanism would fit in.

B. SV Pricing Environment

We consider SVs adopting a product pricing mechanism that is based on the logical network of its consumers and their corresponding security investments. The purpose of an SV to price products in this way is to make additional profits to cover up for the costs of providing insurance to clients as well as make strictly positive profits at all times as an insurer. It has been stated in [6][7] that cyber-insurance with premium discrimination my not lead to the insurer making strictly positive profits at all times. An important question that arises here is: *why would an SV need to price differently based on consumer logical network and his security investment amount when it could transfer a part of his current profits to the insurance business and always make strictly positive profits?* The answer to this question has three parts: (i) a rational firm would not mind finding a way to make more profits than their current scenario, and from principles of basic economics, pricing based on increased client information results in more profits [8], (ii) the amount of security investments of users results in unpaid positive externalities for other users in the logical network and these externalities need to be accounted for in some manner to ensure a certain price fairness amongst consumers[5], and (iii) specifically when an SV is the cyber-insurer, our proposed pricing philosophy would allow appropriate contracts (with fines/rebates) to be handed out to consumers buying insurance. For example, a consumer who generates a high amount of externality would be priced less for SV products than a consumer generating a low amount of externality, and as a result the latter might end up paying a fine, whereas the fomer consumer might get a rebate on his premium.

C. Defining The Model

We assume that each SV (seller) has a set of clients N (consumers) connected via a logical network and using self-defense products manufactured by the SV. Each consumer $i \,\epsilon\, N$ has an utility function, $u_i(\cdot)$, which is given as

$$u_i(x_i, \overrightarrow{x_{-i}}, p_i) = \alpha_i x_i - \beta_i x_i^2 + x_i \cdot \sum_j h_{ij} x_j - p_i x_i, \quad (1)$$

where x_i is the amount[6] of non-negative self-defense goods (manufactured by the SV) consumed by user i, $\overrightarrow{x_{-i}}$ is the vector of investments of users other than i, and p_i is the price charged by the SV to user i per unit of good consumed. Here p_i is the equilibrium market price set by the SV after competing with other SVs in the security product business. $\alpha_i, \beta_i, h_{ij}$ are constants. α_i and β_i can be thought of as indirect representative parameters of advertised cyber-insurance contracts, whereas h_{ij} is the amount of externality user j exerts on user i through his per unit investments. Here $h_{ij} \geq 0$ and $h_{ii} = 0, \forall i$. x_i is assumed to be continuous for analysis tractability reasons. The first and second term in the utility function denotes the utility to a user solely dependent on his own investments, the third term is the positive externality effects of investments made by other users in the network on user i, and the fourth term is the price user i pays for consuming x_i units of self-defense goods manufactured by the SV. We assume here that x_i is bounded. The quadratic nature of the utility function allows for a tractable analysis and a nice second-order approximation of concave payoffs.

The SV accounts for the strategic investment behavior (after it would have set its prices) of users it provides service to, and decides on an optimal pricing scheme that arises from the solution to the following optimization problem.

$$max_{\overrightarrow{p}} \sum_i p_i x_i - c x_i,$$

where $\overrightarrow{p}$ are the vectors of prices charged by the SV to its consumers, and x_i is the amount of self-defense good consumed by consumer i after the SV sets its prices. c is the constant marginal cost to the SV to manufacture a unit of any of its products.

III. Static Pricing Strategy

In this section we first describe our two-stage pricing game between an SV and its consumers. We then state the results of the pricing game.

A. Two-Stage Pricing Game Definition

The game has the following two steps.

1) The SV chooses a price vector $\overrightarrow{p}$ so as to maximize its profits via the following optimization problem.

$$max_{\overrightarrow{p}} \sum_i p_i x_i - c x_i,$$

   where $\overrightarrow{p}$ is price vector charged by the SV to its consumers, per unit of investment, and x_i is the amount of self-defense good consumed (invested) by user i after the SV sets its prices. We consider three types of consumer pricing scenarios in the paper: (i) ***Scenario 1*** - here, the SV does not price discriminate amongst its consumers and all elements of $\overrightarrow{p}$ are identical, i.e., $p_i = p, \forall i$, (ii) ***Scenario 2*** (binary pricing) - here, the SV charges two types of prices per unit of user investment: a regular price denoted as p_{reg} for each user in a particular category, and a discounted price, denoted as p_{dsc} for other users, and (iii) ***Scenario 3*** - here, the SV charges different prices to different consumers and the elements of $\overrightarrow{p}$ are non-identical. c is the constant marginal cost to the SV to manufacture a unit of any of its products.
2) Consumer i chooses to consume x_i units of self-defense products, so as to maximize his utility $u_i(x_i, , \overrightarrow{x_{-i}}, p_i)$ given the prices chosen by the SV.

Since the game consists of two stages, we will analyze the subgame perfect Nash equilibria of this game, instead of just focussing on simple Nash equilibria.

[5]A SV can get investment related information from its insurance clients (and hence estimate externalities) through disclosure agreements signed between the client and the SV as part of some mandate imposed by the government [3].

[6]We assume a continuous amount variable for purposes of analysis. In reality SV products are bundled in a package which is priced as a single item.

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B. Results

In this section we state the results related to our static pricing strategy. We first comment on the equilibrium of the second stage of the two-stage pricing game, *given* a vector of prices $\overrightarrow{p}$. Given $\overrightarrow{p}$, the second stage of our pricing game is a subgame and we denote it as G^{sub}. We the have the following theorem. The proof of the theorem is in the Appendix.

Theorem 1. *G^{sub} has a unique Nash equilibrium and is represented in closed form as*

$$x_i = BR(\overrightarrow{x_{-i}}) = \frac{\alpha_i - p_i}{2\beta_i} + \frac{1}{2\beta_i}\sum_{j\epsilon N} h_{ij}x_j, \tag{2}$$

where $BR(\overrightarrow{x_{-i}})$ is the best response of user i when other users in the network consume $\overrightarrow{x_{-i}}$. In the case when SV does not price discriminate its consumers, the Nash equilibrium vector of user investments is given by

$$\overrightarrow{x} = (Q - G)^{-1}(\overrightarrow{\alpha} - p\overrightarrow{1}), \tag{3}$$

where p is the optimal per unit investment price charged by the SV to all its consumers, and matrix Q takes values $2\beta_i$ at location (i, j) if $i = j$ and zero otherwise.

Theorem Intuition and Implications: The intuition behind a unique Nash equilibrium is the fact that increasing one's consumption incurs a positive externality on his peers, which further implies that the game involves *strategic complementarities*[7] [8], and therefore the equilibria are ordered. This monotonic ordering results in a unique NE. The unique equilibrium implies that the SV just needs to be concerned about the single equilibrium vector of user consumption amounts and base its optimal strategy on that equilibrium vector. If there would be multiple equilibria to G^{sub}, it would be cumbersome for the SV to decide on its optimal strategy. Why? because it would be difficult for non-cooperative users to decide in the first place which equilibrium is the best and then jointly play the best equilibria. As a result the SV might not be sure that the best equilibrium would be played by the users. However, if the SV is able to compute the best equilibria, it could base its pricing strategy on that one irrespective of the equilibria played by the users.

We now discuss the optimal pricing strategy for the SV given that the users self-protect according to the Nash equilibrium of G^{sub}. Before going into the details we first define the concept of a Bonacich centrality in a network of heterogenous users. The Bonacich centrality measure [4] is a sociological graph-theoretic measure of network influence. It assigns relative influence scores to all nodes in the network based on the concept that connections to high-scoring nodes contribute more to the score of the node in question than equal connections to low-scoring nodes. In our work, the Bonacich measure of a user reflects his influence on other users of the network via the externalities generated by him through his self-defense investments. Formally, let G be a matrix defining the logical network of N users (consumers), and having in its entries the h_{ij} values. Let D be a diagonal matrix, and $\overrightarrow{w}$ be a weight vector. The weighted Bonacich centrality vector is given by

$$B(G, D, \overrightarrow{w}) = (I - GD)^{-1}\overrightarrow{w}, \tag{4}$$

where $(I - GD)^{-1}$ is well-defined and non-negative.

We now have our first result regarding the optimal prices charged by the SV to its consumers. *The result addresses pricing scenarios 1 and 3 together as Scenario 1 is a special case of Scenario 3.* The proof of theorem is in the Appendix.

[7]In economics and game theory, the decisions of two or more players are called strategic complements if they mutually reinforce one another.

Theorem 2. *The optimal price vector $\overrightarrow{p}$ charged by the SV is given by*

$$\overrightarrow{p} = \frac{\overrightarrow{\alpha} + c\cdot\overrightarrow{1}}{2} + GQ^{-1}B(G', Q^{-1}, \overrightarrow{w'}) - G^TQ^{-1}B(G', Q^{-1}, \overrightarrow{w'}), \tag{5}$$

where $G' = \frac{G+G^T}{2}$ and $\overrightarrow{w'} = \frac{\overrightarrow{\alpha} - c\cdot\overrightarrow{1}}{2}$.

In the case when the SV does not price discriminate its consumers, the optimal price (same for every consumer) charged per consumer is given by

$$p = \frac{1}{2}\frac{\overrightarrow{1}^T(Q - G)^{-1}(\overrightarrow{\alpha} + c\overrightarrow{1})}{\overrightarrow{1}^T(Q - G)^{-1}\overrightarrow{1}} \tag{6}$$

.

Theorem Intuition and Implications: The optimal price vector in the no price discrimination case is independent of individual node centralities, whereas in the price discrimination case the optimal price vector depends on the Bonacich centrality of individual users. The intuition behind the result is the fact that users tend to invest in security mechanisms proportional to their Bonacich centrality (and in turn generate proportional amount of network externalities) in the Nash Equilibrium [11][12]. Therefore it makes sense for the SV to charge users based on their Bonacich centralities when price discrimination is possible.

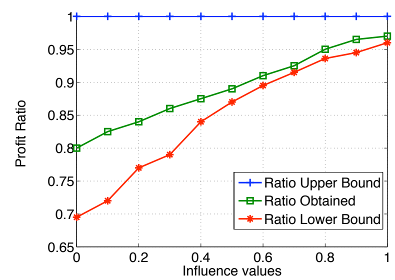

Fig. 1. Profit Ratio and Its Bounds

We now state the following result regarding profit amounts made by a SV from its cyber-insurance business for pricing scenarios 1 and 3. The proof of the theorem is in the Appendix.

Theorem 3. *The profits made by an SV from its cyber-insurance business when the latter does not (does) account for user investment externalities are given by*

$$P_0 = \left\{\left(\frac{\overrightarrow{\alpha} - c\cdot\overrightarrow{1}}{2}\right)^T (Q - G)^{-1}\left(\frac{\overrightarrow{\alpha} - c\cdot\overrightarrow{1}}{2}\right)\right\} \tag{7}$$

and

$$P_1 = \left\{\left(\frac{\overrightarrow{\alpha} - c\cdot\overrightarrow{1}}{2}\right)^T (Q - G')^{-1}\left(\frac{\overrightarrow{\alpha} - c\cdot\overrightarrow{1}}{2}\right)\right\}, \tag{8}$$

.

We plot the ratio of $\frac{P_0}{P_1}$ for *preferential attachment* (PA) graphs in Figure 1. We choose preferential attachment graphs as they represent social/logical network interactions. For networks of size 100, we generate 50 PA graphs for each different value of μ ranging from 0 to 1, where μ reflects the influencing nature of a user in a PA graph w.r.t. the positive externality effects of his security investments made. A μ value of 1 indicates that the user is influenced by all his neighbors but does not influence any one of them, whereas a μ value of zero indicates the opposite. Each point in the plot is the average of the 50 $\frac{P_0}{P_1}$ values obtained per value of μ. Our plot results show

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that the provable *profit ratio bounds are quite tight and are less than 1*[8], implying the fact that an SV does not do that badly in terms of profit when it is not informed about consumer externality values and their network location properties, compared to the case when it does have full information.

Theorem Intuition and Implications: As observed from the plot in Figure 1, the profits to the SV are greater when it accounts for externalities than when it does not, and an SV could make up to 25% extra profits with complete information. This is intuitive in the sense that the SV has more user information when knowing about the externalities and can price optimally to increase its profits. However, in reality it is difficult to measure/observe the externalities. Thus, in spite of getting topological information from the insurer, an SV might have to price its products without taking externalities into account. The profits for the non price discrimination scenario is encapsulated as a special case of P_1 when G has all entries equal except the zero diagonal entries. *We also emphasize here that even in the absence of complete information, partial information will boost SV profits in a proportional manner.*

General Remarks: In practice, it is really difficult to compute the h_{ij} values. One can at best approximate or stochastically estimate it. The Bonacich centrality measure can however be exactly computed given the social network structure of consumers. As mentioned before, even partial information on consumer investments and related externalities will improve profit margins for SVs. In the case a logical network consists of disjoint large subnetworks or sparsely overlapping networks, insurance sellers (SVs) would segment the market at equilibrium and exercise monopoly pricing power in their respective localities. When networks are considerably overlapping it would be difficult for any SV to exercise monopoly pricing of its products. However, in reality there is some heterogeneity between product types of sellers which leads to one being more popular than others, and as a result sellers could have a slight pricing power over their consumers.

The case for two prices (binary pricing): In reality, charging multiple different prices to various consumers may not be very practical to implement. To make things simpler, a SV can opt to charge two types of prices for two different classes of consumers: (i) a discounted price, p_{dsc}, for consumers who have significant positive influence on the security of a network based on their network location and the amount of investments made, (ii) and a regular price, p_{reg} for the other consumers. Thus, the first goal of an SV is to determine the subset of consumers that should be offered the discounted price so as to maximize its own profits.

Given that p_{reg} and p_{dsc} are exogenously specified, the profit optimization problem for an SV is given by

$$Maximize\ (\overrightarrow{p} - c\,\overrightarrow{1})^T (Q-G)^{-1} (\overrightarrow{\alpha} - \overrightarrow{p})$$

$$s.t.\ p_i \,\epsilon\, \{p_{reg}, p_{dsc}\},\ \forall i \,\epsilon\, N.$$

Note here that the expression, $(Q-G)^{-1}(\overrightarrow{\alpha} - \overrightarrow{p})$, in the objective function is the NE investment amount of users in self-defense mechanisms. Thus, we have a combinatorial optimization problem for maximizing the profits of an SV. In order to investigate the tractability of the problem, we formulate it in the following manner:

$$OPT:\ Maximize\ (\delta\overrightarrow{y} + c'\overrightarrow{1})^T (Q-G)^{-1} (\overrightarrow{\alpha'} - \delta\overrightarrow{y})$$

$$s.t.\ y_i \,\epsilon\, \{-1, 1\},\ \forall i \,\epsilon\, N.$$

Here $\delta = p_{reg} - p_T$, where $p_T = \frac{p_{reg} + p_{dsc}}{2}$, $\overrightarrow{a'} = \overrightarrow{a} - p_T$, and $c' = p_T - c \geq \delta$. Note that using these variables, the feasible price allocation can be expressed as $\overrightarrow{p} = \delta\overrightarrow{y} + p_T$. Our next result comments on the intractability of solving OPT. The proof of the result which is in the Appendix, is based on the reduction of OPT from the well-known MAX-CUT problem [5].

[8]Here '1' is the trivial upper bound.

Theorem 4. *Given that p_{reg} and p_{dsc} are exogenously specified in the binary pricing case, an SV's profit optimization problem, OPT, is NP-Hard.*

We plan to design approximation schemes to the optimal solution as part of future work.

IV. Dynamic Pricing Strategy

In the previous section, we analyzed the case when there is just one round of pricing game played between the consumers and the SV. In this section we propose a sequential dynamic pricing strategy for a security vendor, where the consumers and the SV interact in multiple rounds. The motivation for considering multiple rounds is that security products often have different versions coming out at periodic time intervals, and given our framework, the consumers would want take into account past actions of themselves and others in the network, i.e., investment amounts, and modify their actions in the present. Likewise the SV would want to optimally price its product accounting for its consumers' present behaviors. *Our goal is to study whether playing multiple rounds of the two-stage pricing game results in more profits for an SV.*

A. Pricing Model

We assume that the SV adopts a discrete-time sequential pricing strategy where the price vector is changed once every round. The SV approaches each consumer in a particular order per round. Formally, let $\phi^k := \{\phi_1^k, ..., \phi_{n=|N|}^k\}$ be the adopted order of approaching consumers in round k for the SV. Here $\phi_i^k \,\epsilon\, N$, and $\phi_i^k \neq \phi_j^k$ for any $i \neq j$. Let p_i^k be price charged by the SV to consumer i in round k, and x_i^k be the security good consumption by user (consumer) i in round k.

Regarding a consumer's information on other consumer investments prior to round k, we simplistically assume he has accurate knowledge of the quantities consumed by other users in the network prior to his own purchase in round k. Let I_i^k be the information available to user (consumer) i in round k. Then we could mathematically represent it as

$$I_i^k = \left(\overrightarrow{x_i^{(1:k-1)}}, \overrightarrow{p_i^{(1:k-1)}}, \mathbf{X}_{ij}, \overrightarrow{x_j^k} | j \,\epsilon\, N_i^k\right),$$

where $N_i^k \subset N$ is the set of consumers that purchase self-defense mechanisms before consumer i in round k, $\overrightarrow{x_i^{(1:k-1)}}$ is the $k-1$ dimensional vector of investments of consumer i from rounds 1 to $k-1$, $\overrightarrow{p_i^{(1:k-1)}}$ is the $k-1$ dimensional vector of per unit of investment prices charged to consumer i from rounds 1 to $k-1$, $\mathbf{X}_{ij}$ is a 2-dimensional matrix storing the investments of each of N users from rounds 1 to $k-1$, and $\overrightarrow{x_j^k}$ is the investment made by consumer j in round k. Let $y_i^k = \sum_{j=1}^k x_i^k$ be the cumulative self-defense quantity consumed by user i up to round k. The utility function of consumer i in round k is then given by

$$u_i(x_i^k, I_i^k, p_i^k) = A - B + C + D - E, \qquad (9)$$

where

$$A = \alpha_i \left(y_i^{k-1} + x_i^k\right),$$

$$B = \beta_i \left(y_i^{k-1} + x_i^k\right)^2,$$

$$C = \left(y_i^{k-1} + x_i^k\right) \sum_{j\,\epsilon N} h_{ij} y_j^{k-1},$$

$$D = \left(y_i^{k-1} + x_i^k\right) \sum_{j\,\epsilon N_i^k} h_{ij} x_j^{k-1},$$

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and

$$E = \sum_{j=1}^{k} p_i^j x_i^j,$$

The terms A, B, C, D, and E have the same interpretation as the corresponding terms of Equation 1, except that now we are considering cumulative investments up to round k, instead of just investments in one round, as that in Equation 1. For each round k, consumer i will look to maximize $u_i(x_i^k, I_i^k, p_i^k)$, whereas the SV will choose $\{p_i^k\}$ to maximize $\sum_{i \, \epsilon \, N} p_i^k x_i^k - cx_i^k$. We thus have the following optimization problem, the solution to which gives the *optimal ordering* that maximizes the SV profits in round k.

$$argmax_{\gamma^k \, \epsilon \, Perm(\phi^k)} P^k(\gamma^k)$$

where $Perm(\phi^k)$ is the set of all permutations of the elements of ϕ^k, i.e., the set of all possible orderings of consumers in round k, and $\gamma^k := \{\gamma_1^k, ..., \gamma_n^k\}$ be the adopted *optimal* order of approaching consumers in round k that maximizes the objective function. Note that $P^k(\cdot)$ stands for the optimal solution to the SV profits in round k *given* any particular order. It is the solution to another optimization problem in itself, where we are required to find the optimal user investment amounts that maximize SV profits given any particular order, ϕ^k. We formulate the optimization problem as follows:

$$P^k(\phi^k) = max \sum_{i=1}^{n} p_{\phi_i^k}^k x_{\phi_i^k}^k - cx_i^k$$

$$s.t. \; x_{\phi_i^k}^k = argmax_{z \geq 0} u_{\phi_i^k}\left(z, \tilde{I}_{\phi_i^k}^k, p_{\phi_i^k}^k\right), \forall i,$$

where $\tilde{I}_i^k = \left(\overrightarrow{\tilde{x}_i^{(1:k-1)}}, \overrightarrow{\tilde{p}_i^{(1:k-1)}}, \tilde{\mathbf{X}}_{ij}, \overrightarrow{\tilde{x}_j^k} | j \, \epsilon \, N_i^k\right)$ is the information available to consumer ϕ_i^k in round k. $\tilde{x}_i^k$ and $\tilde{p}_i^k$ are the optimal consumed amount of security good and price of consumer i in round k respectively.

B. Results

Note that in order to solve the profit maximization problem for the SV, in each round we need to consider $n!$ possible orderings to find the optimal order, which makes the state space size intractable for analysis. For tractability of analysis, we assume that the h_{ij} values are symmetric and $c = 0$. We now state two results (theorems) on the outcome of the pricing game played for multiple rounds. The proofs of these theorems are in the Appendix.

Theorem 5. *The optimal solution to $\overrightarrow{\tilde{x}^k(\phi^k)}$ and the corresponding $P^k(\phi^k)$ value are unique, and independent of the ordering in ϕ^k. The closed form of the optimal solution for $\overrightarrow{\tilde{x}^k(\phi^k)}$ in each round k is given by*

$$\overrightarrow{\tilde{x}^k(\phi^k)} = [2Q - G]^{-1}[I - (Q - G)(2Q - G)^{-1}]^{k-1}\overrightarrow{\alpha}, \quad (10)$$

where matrix Q takes values $2\beta_i$ at location (i, j) if $i = j$ and zero otherwise.
The optimal per unit consumption prices charged by the SV to each consumer i in round k is given by

$$\tilde{p}_i^k = \alpha_i - 2\beta_i \tilde{y}_i^{k-1} + \sum_{j \, \epsilon \, N} h_{ij}\tilde{y}_j^{k-1} - 2\beta_i \tilde{x}_i^k + \sum_{j<i} h_{ji}\tilde{x}_j^k. \quad (11)$$

Theorem Intuition and Implications: The independence of the optimal consumer investments with respect to consumer ordering follows from the fact that h_{ij} values are symmetric. We also observe that the optimal consumption amounts and the prices charged by the SV do not depend on the centrality measures. This is again due to the fact the h values are symmetric amongst users (consumers). Externality effects rely on the topological structure - thus it is evident that with symmetric externality effects, topology does not affect the optimal consumption and price structure. However, in reality there will be asymmetric externality effects, and we conjecture that the optimal consumer consumption and SV prices will depend on the location of network users.

We now have the following theorem related to the comparison of SV profits between the static pricing strategy and the dynamic pricing strategy cases.

Theorem 6. *The profit obtained by an SV via our proposed dynamic pricing strategy strictly dominates that obtained due to the static pricing strategy (even if the strategy prices uniformly for all consumers).*

Theorem Intuition and Implications: The theorem follows from the fact that an SV has more information from consumer investment history and prices its consumers accordingly to increase profits. In this work we have assumed symmetric externality effects in the dynamic pricing case, which is equivalent to having partial information about consumer investments. We observe that in the absence of complete information on consumer investment history, an SV can make improved profits with partial information when compared to the static pricing case. With more information on consumer investments, SVs can only increase their profits, and accordingly allocate some profits to its cyber-insurance business.

C. Consumer Fairness Through Dynamic Pricing

We have already shown in the dynamic pricing setting above that the order in which a consumer is approached by a SV does not have an impact on his optimal pricing strategy, under symmetric externality effects. In this case we can explore this *invariance of ordering* property to ensure fairness amongst the network users. By the term 'fairness' we mean distributing the net utility of all consumers in a network in a fair manner over time. Fairness is an important parameter from a consumer perspective. An SV could increase its reputation and consumer popularity (and in turn sales) by ensuring fairness amongst its network users. In this paper we consider the *max-min fairness* allocation concept (not relevant for uniform consumer pricing). The goal of the SV is to choose the consumer visit order in each round so as to achieve the max-min fairness goal over time. We emphasize again that irrespective of the visit order, the SV keeps its optimal profits fixed.

Consider the dynamic pricing strategy where the SV approaches its consumers in a same fixed order every round. Assume the order to be $\{1, 2, ..., n = |N|\}$. The price in round k, charged to a consumer i, he being the mth person visited by the SV, is given by

$$\tilde{p_i^k} = a^{k-1}\frac{\alpha(2\beta - (n-m)h)}{4\beta - (n-1)h}, \forall i \, \epsilon \, N, \quad (12)$$

where a is given by

$$a = \frac{2\beta}{4\beta - (n-1)h} < 1. \quad (13)$$

Note that in case of a symmetric graph $\overrightarrow{\alpha} = \alpha$, and $\overrightarrow{\beta} = \beta$. It is evident from the equation that prices charged to consumers increase linearly in the order of visit. This generates a wide disparity in the individual user utility with the nth visited consumer having the least utility in all rounds. Thus we do not get a fair net utility allocation amongst the consumers over time.

To alleviate the fairness issue, we make the following simple change in the dynamic pricing protocol: the SV chooses any arbitrary ordering (assume $\{1, 2, ..., .n\}$) in the first round, but uses a max-min fairness criteria to find the order in subsequent rounds. For a symmetric graph, for each $k > 1$, we choose the ordering $\phi^k = \{\phi_1^k, ..., \phi_n^k\}$ such that the following holds.

$$\phi_i^k = argmin_{i \, \epsilon \, N - \{\phi_1^k, ..., \phi_{i-1}^k\}} u_i(x_i^k, I_i^k, p_i^k)$$

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We then obtain the net utility of a consumer visited in the mth position in any round k as

$$u_m(x_m^k, I_m^k, p_m^k) = \frac{a^2\alpha^2}{4\beta^2}\left((2\beta-(n-m)h)+\frac{(2\beta-mh)a^2}{1-a^2}\right). \quad (14)$$

This equation implies that any other order in round $k>1$ in which consumer n is not scheduled in the beginning will lead to him having a lower net utility compared to the orderings in which he is scheduled in the beginning. Thus, we achieve a max-min fairness over time, in regard to allocating net utility of consumers.

V. Conclusion and Future Work

Cyber-insurance markets, be it monopolistic, competitive, or oligopolistic might not turn out to be very profitable for cyber-insurer/s. This is more so the case because these markets are applicable to interdependent and correlated risk environments. In such environments, insurers in the worst case could go bankrupt, and generally might make zero or non-negative expected profits even under client contract discrimination, in the process of sustaining a certain level of social welfare requirement from a regulatory agency such as the government. To alleviate this problem a security vendor can be a part of the cyber-insurance driven security ecosystem by being the insurer itself and channeling the extra profits obtained from its security product business to its cyber-insurance business. Strict positive profits at all times for a cyber-insurer will satisfy its interest in an insurance-driven security ecosystem and result in more than moderate success of cyber-insurance markets. The channeling process can be made possible by SVs by getting client information on their logical network location as well as the amount of externalities caused due to their security investments. According to basic micro-economic theory, pricing techniques based on such additional client information (be it perfect or imperfect) generates extra profits for SVs compared to their traditional pricing methods. In the process, an SV could also ensure a lock-in effect amongst its insurance clients by enforcing the latter to buy security products only from his SV, in turn increase SV demand for security products. In this paper we have shown that (i) price discriminating consumers in proportion to the Bonacich centrality of individual users results in maximum profits for an SV, (ii) an SV could make up to 25% additional profits with perfect client information, (iii) the problem of price discriminating consumers in order to maximize SV profits when there are only two price categories, i.e., regular and discounted, is NP-Hard. We have also showed that a pricing game played between consumers and an SV for multiple rounds can result in greater maximum profits for both, the SV and an insurer when compared to the outcome of the pricing game played for just one round. Finally, for the dynamic pricing game, we showed that for a given optimal pricing strategy that is invariant of the order in which the SV visits its consumers, it is possible for the SV to allocate the latters' net utility in a max-min fair manner over time by changing the order of visits in each round. Maintaining the same order of consumer visits would result in optimal profits for the SV but would not be fair to certain users who would always (in every round) be visited near the end and accrue lower utilities than their counterparts who were visited earlier.

As part of future work we plan to design an approximation algorithm that provides a nice approximation guarantee to the binary pricing problem. Note that in this paper, we assumed the binary prices p_{reg} and p_{dsc} to be exogenously specified for the binary pricing problem. In this regard, we also aim to compute the optimal binary prices that maximizes SV profits when they are not exogenously specified. We also aim to extend our dynamic pricing strategy to account for asymmetric externality effects of consumer investments.

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VII. Appendix

In this section, we prove Theorems 1-6.

Proof of Theorem 1. The proof relies on the results on the following lemmas. We first state and prove the relevant lemmas required for the proof of Theorem 1 and follow it up with the proof of the main theorem.

Lemma 1. *The game G^{sub} is supermodular*[9].

Proof. The payoff/utility functions are continuous, the strategy sets are compact subsets of real space, and for any two consumers $i, j \,\epsilon N$, $\frac{\partial^2 u_i}{\partial x_i \partial x_j} \geq 0$. Hence G^{sub} is supermodular. ■

Lemma 2. The spectral radius of $Q^{-1}G$ is smaller than 1, and the matrix $I - Q^{-1}G$ is invertible.

Proof. Let $\overrightarrow{v}$ be an eigenvector of $Q^{-1}G$ with λ being the corresponding eigenvalue, with $|v_i| > |v_j|$ for all $j \,\epsilon N$. We have the following equation due to the fact that $(Q^{-1}G)\overrightarrow{v} = \lambda\overrightarrow{v}$.

$$|\lambda v_i| = |(Q^{-1}G_i)\overrightarrow{v}| \leq \sum_{j\,\epsilon\,N}(Q^{-1}G)_{ij}|v_j| \leq \frac{1}{2\beta_i}|v_i|\sum_{j\,\epsilon\,N}h_{ij} < \frac{v_i}{2}. \quad (15)$$

Here $(Q^{-1}G)_i$ denotes the $i-th$ row of $(Q^{-1}G)$. Since the equation holds for any eigenvalue-eigenvector pair, the spectral radius of $(Q^{-1}G)$ is strictly smaller than 1. Now observe that each eigenvalue of $I - Q^{-1}G$ can be written as $1-\lambda$. Since the spectral radius of $Q^{-1}G$ is strictly smaller than 1, none of the eigenvalues of $I - Q^1G$ is zero, and thus the matrix is invertible. ■

Now we continue with the proof of Theorem 1. Since G^{sub} is a supermodular game, the equilibrium set has a minimum and a maximum element [13]. Let $\overrightarrow{x}$ denote the maximum of the equilibrium set and let S be such that $x_i > 0$ only if $i \,\epsilon S$. If $S = \phi$ there cannot be another equilibrium point, since $\overrightarrow{x} = 0$ is the maximum of the equilibrium set. Assume for a contradictory purpose that $S \neq \phi$ and there is another equilibrium $\overrightarrow{\tilde{x}}$, of the game. By the supermodularity property of G^{sub}, $x_i \geq \tilde{x}_i, \forall i \,\epsilon N$. Allow k to equal $argmax_{i\,\epsilon\,N} x_i - \tilde{x}_i$. Since $\overrightarrow{x}$ and $\overrightarrow{\tilde{x}}$ are not equal, we have

[9] In supermodular games, the marginal utility of increasing a player's strategy increases with the increases in the other players' strategies.

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$x_k - \tilde{x_k} > 0$. Since at NE no consumer has an incentive to increase or decrease his consumption, we have

$$x_k - \tilde{x}_k \leq \frac{1}{2\beta_k} G_k(\overrightarrow{x} - \overrightarrow{\tilde{x}}) = \frac{1}{2\beta_k} \sum_j h_{kj}(x_j - \tilde{x_j}), \tag{16}$$

where G_k is the $k-th$ row of G. But we have

$$\frac{1}{2\beta_k} \sum_j h_{kj}(x_j - \tilde{x_j}) \leq \frac{x_k - \tilde{x}_k}{2\beta_k} \sum_j h_{kj} < x_k - \tilde{x_k} \tag{17}$$

Thus, we reach a contradiction and G^{sub} has a unique Nash equilibrium. ■

Proof of Theorem 2. We have from Lemma 2 that $Q - G$ is non-singular and as a result the following equation holds.

$$\overrightarrow{p} = \overrightarrow{\alpha} - (Q-G)\left(Q - G - \frac{G^T - G}{2}\right)^{-1} \frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2} \tag{18}$$

Equation 18 can we rewritten as

$$\overrightarrow{p} = \overrightarrow{\alpha} - \left(I - \frac{G^T - G}{2}(Q-G)^{-1}\right)^{-1} \frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2} \tag{19}$$

By the *matrix inversion lemma* [9], we have

$$\left(I - \frac{G^T - G}{2}(Q-G)^{-1}\right)^{-1} = I + \frac{G^T - G}{2}\left(Q - \frac{G^T + G}{2}\right)^{-1} \tag{20}$$

Thus, from Equation 19 it follows that

$$\overrightarrow{p} = \frac{\alpha + c\overrightarrow{1}}{2} - \frac{G^T - G}{2}\left(Q - \frac{G^T + G}{2}\right)^{-1} \frac{\overrightarrow{\alpha} - c\overrightarrow{1}}{2} \tag{21}$$

Applying Equation 21 and using the definition of weighted Bonacich centrality, we get

$$\overrightarrow{p} = \frac{\overrightarrow{\alpha} + c \cdot \overrightarrow{1}}{2} + GQ^{-1}B(G', Q^{-1}, \overrightarrow{w'}) - G^T Q^{-1} B(G', Q^{-1}, \overrightarrow{w'})$$

and thus prove Theorem 2. ■

Proof of Theorem 3. The optimal price vector of the SV without and with the consideration of externality effects are given by the following equations.

$$\overrightarrow{p_0} = \frac{\overrightarrow{\alpha} + c \cdot \overrightarrow{1}}{2}. \tag{22}$$

and

$$\overrightarrow{p_1} = \overrightarrow{\alpha} - (Q-G)\left(Q - \frac{G + G^T}{2}\right)^{-1} \frac{\alpha - c \cdot \overrightarrow{1}}{2}. \tag{23}$$

The corresponding consumption vectors are given by

$$\overrightarrow{x_0} = (Q-G)^{-1} \frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}. \tag{24}$$

and

$$\overrightarrow{x_1} = (Q-G')^{-1} \frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}. \tag{25}$$

It then follows that

$$P_0 = (\overrightarrow{p_0} - c \cdot \overrightarrow{1})^T \overrightarrow{x_0} = \frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}(Q-G)^{-1} \frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}. \tag{26}$$

and

$$P_1 = (\overrightarrow{p_1} - c \cdot \overrightarrow{1})^T \overrightarrow{x_1}, \tag{27}$$

P_1 can be re-written as

$$P_1 = X - Y,$$

where

$$X = 2\left(\frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}\right)^T \left(\frac{R + R^T}{2}\right)^{-1} \left(\frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}\right),$$

and

$$Y = \left(\frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}\right)^T \left(\frac{R + R^T}{2}\right)^{-1} \left(\frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}\right).$$

Thus, we have

$$P_1 = \left\{\left(\frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}\right)^T (Q - G')^{-1} \left(\frac{\overrightarrow{\alpha} - c \cdot \overrightarrow{1}}{2}\right)\right\},$$

and we prove our theorem. ■.

Proof of Theorem 4. The well known MAX-CUT problem [5] is as follows:

$$max \sum_{(i,j)\,\epsilon\, E} W_{ij}(1 - x_i x_j)$$

$$s.t.\ x_i\,\epsilon\,\{-1, +1\},\ \forall i\,\epsilon\, V,$$

where W denotes a matrix of binary weights consisting of 0s and 1s. The solution to this problem corresponds to a cut as follows: let S be the agents who were assigned value 1 in the optimal solution. Then it is straightforward to see that the value of the objective function corresponds to the size of the cut defined by S and $V - S$. The problem can also be re-wriiten as

$$P0:\ min\ \overrightarrow{x}^T W \overrightarrow{x}$$

$$s.t.\ x_i\,\epsilon\,\{-1, +1\},\ \forall i\,\epsilon\, V.$$

Now consider the following related problem:

$$P1:\ min\ \overrightarrow{x}^T W \overrightarrow{x}$$

$$s.t.\ x_i\,\epsilon\,\{-1, +1\},\ \forall i\,\epsilon\, V,$$

where W is a symmetric matrix with rational entries that satisfy $0 < W^T = W < 1$. In the proof of Theorem 4, we will first show that $P1$ is NP-Hard by reducing from MAX-CUT. We will then reduce P1 to OPT to claim the correctness of our theorem.

Lemma 3. *P1 is NP-Hard.*

Proof. We prove our claim by reducing P1 from P0. Let W be the weight matrix in an instance of P0. Let $W_\epsilon = \frac{1}{2}(\epsilon + W)$, where ϵ is a rational number between 0 and $\frac{1}{2n^2}$ and $|V| = n$. Observe that for any feasible $\overrightarrow{x}$ in P0 or P1, it follows that

$$2\overrightarrow{x}^T W_\epsilon(\overrightarrow{x}) - n2\epsilon \leq \overrightarrow{x}^T W \overrightarrow{x} \leq 2\overrightarrow{x}^T W_\epsilon(\overrightarrow{x}) + n2\epsilon.$$

Because the objective of P0 is always an integer and $n^2\epsilon < \frac{1}{2}$, the cost of P0 for any feasible vector $\overrightarrow{x}$ can be obtained from the cost of P1 by scaling and rounding. Therefore, since P0 is NP-Hard, it follows that P1 is also NP-Hard. ■.

Having proved P1 to be NP-Hard, we now prove Theorem 4 by reducing OPT from P1. We consider special instances of OPT where $G = G^T$, $c = 0$, and $\overrightarrow{\alpha} = [\alpha, \ldots\ldots, \alpha]$, and $\alpha = p_{reg} + p_{dsc}$. OPT can then be re-casted as

$$OPT2:\ min\ \overrightarrow{x}^T (Q - G) \overrightarrow{x}$$

$$s.t.\ x_i\,\epsilon\,\{-1, +1\},\ \forall i\,\epsilon\, N.$$

Now consider an instance of P1 with $W > 0$. Note that because $x_i^2 = 1$, P1 is equivalent to

$$min\ \overrightarrow{x}^T (W + \gamma I) \overrightarrow{x}$$

$$s.t.\ x_i\,\epsilon\,\{-1, +1\},\ \forall i\,\epsilon\, V,$$

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where we choose γ as an integer such that

$$\gamma > 4 \cdot max\{\rho(W), \sum_{i,j} \frac{W_{i,j}}{min_{i,j} W_{ij}}\}$$

and $\rho(\cdot)$ is the spectral radius of its argument. The definition of γ implies that the spectral radius of $\frac{W}{\gamma}$ is less than 1. Therefore we have

$$(W+\gamma I)^{-1} = \frac{1}{\gamma}\left(I - \frac{1}{\gamma}(W - \frac{W^2}{\gamma}) - \frac{W^2}{\gamma^3}(W - \frac{W^2}{\gamma}).....\right). \tag{28}$$

Since all entries of W and $\left(W - (\frac{W^2}{\gamma})\right)$ are positive, the above equality implies that the off-diagonal entries of $(W+\gamma I)^{-1}$ are negative. Therefore $(W+\gamma I)^{-1} = (Q-G)$ for some diagonal matrix Q and some $G \geq 0$. Thus, it follows that

$$((Q-G)\overrightarrow{1})_k = \left(\frac{1}{\gamma}\left(I - \frac{W}{\gamma} + \frac{W^2}{\gamma^2}....\right)\overrightarrow{1}\right)_k. \tag{29}$$

Since $W > 0$, we have

$$((Q-G)\overrightarrow{1})_k \geq \frac{1}{\gamma}\left(1 + \left(-\frac{W\overrightarrow{1}}{\gamma}) - \frac{W^2\overrightarrow{1}}{\gamma^2}....\right)_k \overrightarrow{1}\right).$$

From the definition of γ it follows that $\frac{W\overrightarrow{1}}{\gamma} \leq (\frac{(\sum_{i,j} W_{ij})}{\gamma})\overrightarrow{1} \leq \frac{1}{4}\overrightarrow{1}$. The above inequality implies that

$$((Q-G)\overrightarrow{1})_k \geq \frac{1}{\gamma}\left(1 - \frac{1}{4}\left(\sum_{l=0}^{\infty}\left(\frac{1}{4}\right)^l\right)\right) = \frac{1}{\gamma}\left(\frac{2}{3}\right) > 0. \tag{30}$$

Thus, P1 can be reduced to an instance of OPT2 by defining Q and G according to $(W+\gamma I)^{-1} = (Q-G)$. Therefore it follows that OPT2, and hence OPT, are NP-Hard. ■.

Proof of Theorem 5. Consider the first round. We fix and order $\phi^1 = \{\phi_1^1, ..., \phi_n^1\}$. The profit of the SV is given as

$$P^1(\phi^1, \overrightarrow{x^1}) = \sum_i p_{\phi_i^1} x_{\phi_i^1} = \sum_i (\alpha_{\phi_i^1} - 2\beta_{\phi_i^1} x_{\phi_i^1} + \sum_{j<i} h_{\phi_i^1,\phi_j^1} x_{\phi_j^1}) x_{\phi_j^1}. \tag{31}$$

Let $\overrightarrow{\tilde{x}^1}$ be the optimal consumption of user i in round 1. Then for symmetric h values, we have

$$\alpha_i - 4\beta_i \overrightarrow{\tilde{x}_i^1} + \sum_j h_{ij}\tilde{x}_j^1 = 0, \tag{32}$$

which is independent of ϕ. Thus $\overrightarrow{\tilde{x}^1} = (2Q-G)^{-1}\overrightarrow{\alpha}$. The optimal profit for the SV in round 1 is given by

$$P^1(\phi^1, \overrightarrow{x^1}) = \frac{1}{2}\overrightarrow{\alpha}^T(2Q-G)^{-1}\overrightarrow{\alpha}. \tag{33}$$

The results for rounds 2 to k can be proved inductively. We have the following equations for a particular round k

$$P^k(\phi^k, \overrightarrow{x^k}) = \sum_i \Phi x_{\phi_j^k}. \tag{34}$$

where

$$\Phi = (\alpha_{\phi_i^k} - 2\beta_{\phi_i^k}\tilde{y}_{\phi_i^k}^{k-1} + \sum_j h_{\phi_i^k,\phi_j^k}\tilde{y}_{\phi_j^k}^{k-1} - 2\beta_{\phi_i^k} x_{\phi_i}^k + \sum_{j<i} h_{\phi_i^k,\phi_j^k} x_{\phi_j^k})$$

Now since the spectral radius of matrix $[I - (Q-G)(2Q-G)^{-1}]^{k-1}\overrightarrow{\alpha}]$ is less than 1, we have

$$\overrightarrow{\alpha^k} = [I - (Q-G)(2Q-G)^{-1}]^{k-1}\overrightarrow{\alpha}, \tag{35}$$

where

$$\overrightarrow{\alpha^k} = \alpha_{\phi_i^k} - 2\beta_{\phi_i^k}\tilde{y}_{\phi_i^k}^{k-1} + \sum_j h_{\phi_i^k,\phi_j^k}\tilde{y}_{\phi_j^k}^{k-1}.$$

Thus, the optimal consumption amount of user i in round k is given by

$$\alpha_i^k - 4\beta_i\overrightarrow{\tilde{x}_i^k} + \sum_{j<i} h_{\phi_i^k,\phi_j^k}\tilde{x}_{\phi_j^k}^k + \sum_{j>i} h_{\phi_i^k,\phi_j^k}\tilde{x}^k = 0, \tag{36}$$

which is independent of order in ϕ. Thus, we have

$$\overrightarrow{\tilde{x}} = (2Q-G)^{-1}[I - (Q-G)(2Q-G)^{-1}]^{k-1}\overrightarrow{\alpha}. \tag{37}$$

The total consumption as the number of rounds goes to infinity is given as

$$\lim_{k\to\infty}\overrightarrow{\tilde{y}^k} = \sum_{k=1}^{\infty}(2Q-G)^{-1}[I - (Q-G)(2Q-G)^{-1}]^{k-1}\overrightarrow{\alpha}. \tag{38}$$

Simplifying, we get

$$\lim_{k\to\infty}\overrightarrow{\tilde{y}^k} = (Q-G)^{-1}\overrightarrow{\alpha}. \tag{39}$$

Thus, we have proved Theorem 5. ■

Proof of Theorem 6. The asymptotic profit, P_D, obtained by the SV via its dynamic pricing strategy is expressed as the following sequence of equation

$$P_D = \sum_{k=1}^{\infty} P^k(\phi^k, \overrightarrow{x^k}); \tag{40}$$

$$P_D = \frac{1}{2}\sum_{k=1}^{\infty}\overrightarrow{\alpha}^T[(2Q-G)^{-1}Q]^{k-1}(2Q-G)^{-1}[Q(2Q-G)^{-1}]^{k-1}\overrightarrow{\alpha}; \tag{41}$$

$$P_D = \frac{1}{2}\sum_{k=1}^{\infty}\overrightarrow{\alpha}^T Q^{-1}[Q(2Q-G)^{-1}]^{2k-1}\overrightarrow{\alpha}. \tag{42}$$

Note that the sum $\frac{1}{2}\sum_{k=1}^{\infty}\overrightarrow{\alpha}^T Q^{-1}[Q(2Q-G)^{-1}]^{2k-1}\overrightarrow{\alpha}$ converges due to the fact that the spectral radius of $Q(2Q-G)^{-1}]^{2k-1}\overrightarrow{\alpha}$ is less than 1. Thus, the optimal profit obtained by the SV is given as

$$P_D = \frac{1}{2}\overrightarrow{\alpha}^T(Q-G)^{-1}(2I - (Q-G)(2Q-G)^{-1})^{-1}\overrightarrow{\alpha} \tag{43}$$

The optimal profit, P_S, obtained by the SV in the static pricing case is given as

$$P_S = \frac{1}{4}\overrightarrow{\alpha}^T(Q-G)^{-1}\overrightarrow{\alpha}. \tag{44}$$

Thus, the difference in profits of the SV between the dynamic pricing and the static pricing cases is $P_D - P_S$, and is given by

$$P_D - P_S = \frac{1}{4}\overrightarrow{\alpha}^T\left(\frac{3}{2}Q - G\right)^{-1}\overrightarrow{\alpha}. \tag{45}$$

Since the entries in $(\frac{3}{2}Q - G)$ are non-negative, we have $P_D - P_S > 0$. Thus we have proved Theorem 6. ■.

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