Exact Evaluation of Outage Probability in Correlated Lognormal Shadowing Environment

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***Abstract*—Outage probability computation has been extensively studied in cellular radio systems where the interference is modeled by a sum of lognormal random variables. Since the sum of correlated lognormal random variables distribution does not have a closed-form expression, many approximation methods and bounds were proposed in the literature. However the accuracy of each method relies highly on the region of the resulting distribution being examined and the individual lognormal parameters, i.e., mean and variance. There is no such method which can provide the needed accuracy for all cases. This paper proposes a universal yet very simple approximation method for the sum of correlated lognormal random variables based on log skew normal approximation. Hence, the outage probability is accurately computed over the whole range of dB spreads for any correlation coefficient. We show that our method provides same results as Monte Carlo simulation for all cases.**

Keywords—Correlated Lognormal Sum, Log Skew Normal, Interference, Outage Probability.

I. INTRODUCTION

The outage probability represents an important performance metric in cellular system. The Signal-to-Interference-plus-Noise Ratio (SINR) has to be kept above certain threshold to guarantee a certain level of quality of service (QoS). In wireless communication, lognormal shadowing is the dominant contributor to interferences. Outage probability computation based on the sum of interferers requires providing the sum of lognormal Random Variables (RVs) distribution. However, there is no closed-form expression for this distribution yet.

Several methods have been proposed in order to approximate the sum of correlated lognormal RVs. Since numerical methods require a time-consuming numerical integration, which is not adequate for practical cases, we consider only analytical approximation methods. Ref. [1] gives an extension of the widely used iterative method known as Schwartz and Yeh method [2]. Some other resources use an extended version of Fenton and Wilkinson methods [3-4]. These methods are based on the fact that the sum of dependent lognormal distributions can be approximated by another lognormal distribution. The non-validity of this assumption at distribution tails, as shown in [5], is the main raison for its fail to provide a consistent approximation to the sum of correlated lognormal distributions over the whole range of dB spreads. Furthermore, the accuracy of each method depends highly on the region of the resulting distribution being examined. For example, Schwartz and Yeh based methods provide acceptable accuracy in low-precision region of the Cumulative Distribution Function (CDF) (i.e., 0.01–0.99) and the Fenton–Wilkinson method offers high accuracy in the high-value region of the CDF (i.e., 0.9–0.9999). Both methods break down for high values of standard deviations. Ref [6] proposes an alternative method based on Log Shifted Gamma (LSG) approximation to the sum of dependent lognormal RVs. LSG parameters estimation is based on moments computation using Schwartz and Yeh method. Although, LSG exhibits an acceptable accuracy, it does not provide good accuracy at the lower region. In this paper, we propose to use Log Skew Normal (LSN) approximation for outage probability computation in correlated lognormal shadowing environment. Simulations show that our approximation provides exact results for outage probability computation over a wide range of dB spread for any correlation factor.

The rest of the paper is organized as follows: In section 2, a brief description of the lognormal and log skew normal distributions is given. Then we provide LSN parameters derivation procedure in order to approximate the sum of correlated lognormal distributions. In section 3, we use the LSN approximation for outage probability computation. In section 4, we validate our approach based on Monte Carlo simulations results. The conclusion remarks are given in Section 5.

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II. SUM OF CORRELATED LOGNORMALS USING LSN APPROXIMATION

A. Sum of correlated Lognormals

Given X, a Gaussian RV with mean μ_X and variance σ_X^2, then $L=e^X$ is a lognormal RV with Probability Density Function (PDF):

$$f_L(l,\mu_X,\sigma_X)=\begin{cases}\frac{1}{\sqrt{2\pi}l\sigma_X}\exp(-\frac{1}{2\sigma_X^2}\left[\ln(l)-\mu_X\right]^2) & l>0\\ 0 \quad otherwise\end{cases} \tag{1}$$

Usually X represents power variation measured in dB. Considering X_{dB} with mean μ_{dB} and variance σ_{dB}^2, the corresponding lognormal RV $L=10^{\frac{X_{dB}}{10}}$ has the following PDF:

$$f_L(l,\mu_{dB},\sigma_{dB})=\begin{cases}\frac{1}{\sqrt{2\pi}l\sigma_{dB}}\exp(-\frac{1}{2\sigma_{dB}^2}\left[10\log(l)-\mu_{dB}\right]^2) & l>0\\ 0 \quad otherwise\end{cases} \tag{2}$$

Where $\mu_{dB}=\frac{\mu_X}{\xi}$, $\sigma_{dB}=\frac{\sigma_X}{\xi}$

and $\xi=\frac{\ln(10)}{10}$

The first two central moments of L may be written as:

$$m=e^{\mu}e^{\sigma^2/2} \tag{3}$$

$$D^2=e^{2\mu}e^{\sigma^2}(e^{\sigma^2}-1) \tag{4}$$

Correlated Lognormals sum distribution corresponds to the sum of dependent lognormal RVs, i.e:

$$\Lambda=\sum_{j=1}^{N}L_j=\sum_{j=1}^{N}e^{X_j} \tag{5}$$

We define $\vec{L}=(L_1,L_2...L_N)$ as a strictly positive random vector such that the vector $\vec{X}=(X_1,X_2...X_N)$ with $X_j=\log(L_j)$, $1\le j\le N$ has an n-dimensional normal distribution with mean vector $\mu=(\mu_1,\mu_2...\mu_N)$ and covariance matrix M with $M(\mathrm{i},\mathrm{j})=Cov(X_i,X_j)$, $1\le i,j\le N$. $\vec{L}$ is called an n-dimensional log-normal vector with parameters $\vec{\mu}$ and M.

B. Log Skew Normal Distribution

The standard skew normal distribution was firstly introduced in [7] and was independently proposed and systematically investigated by Azzalini [8]. The random variable X is said to have a scalar $SN(\lambda,\varepsilon,\omega)$ distribution if its density is given by:

$$f_X(\mathrm{x};\lambda,\varepsilon,\omega)=\frac{2}{\omega}\varphi(\frac{x-\varepsilon}{\omega})\phi(\lambda\frac{x-\varepsilon}{\omega}) \tag{6}$$

where $\varphi(\mathrm{x})=\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}$, $\phi(\mathrm{x})=\int_{-\infty}^{x}\varphi(\zeta)\,\mathrm{d}\zeta$

With λ is the shape parameter which determines the skewness, ε and ω represent the usual location and scale parameters and φ, ϕ denote, respectively, the PDF and the CDF of a standard Gaussian RV.

The CDF of the skew normal distribution can be easily derived as:

$$F_X(\mathrm{x};\lambda,\varepsilon,\omega)=\phi(\frac{x-\varepsilon}{\omega})-2\,\mathrm{T}(\frac{x-\varepsilon}{\omega},\lambda) \tag{7}$$

Where function $\mathrm{T}(\mathrm{x},\lambda)$ is Owen's T function expressed as:

$$\mathrm{T}(\mathrm{x},\lambda)=\frac{1}{2\pi}\int_0^{\lambda}\frac{\exp\left\{-\frac{1}{2}\mathrm{x}^2(1+\mathrm{t}^2)\right\}}{(1+\mathrm{t}^2)}dt \tag{8}$$

A fast and accurate calculation of Owen's T function is provided in [9].

Similar to the relation between normal and lognormal distributions, given a skew normal RV X then $L=10^{\frac{X_{dB}}{10}}$ is a log skew normal distribution. The CDF and PDF of L can be easily derived as:

$$f_L(l;\lambda,\varepsilon_{dB},\omega_{db})=\begin{cases}\frac{2}{\xi\omega_{db}l}\varphi(\frac{10\log(l)-\varepsilon_{dB}}{\omega_{db}})\phi(\lambda\frac{10\log(l)-\varepsilon_{dB}}{\omega_{db}}) & l>0\\ 0 \quad otherwise\end{cases} \tag{9}$$

$$F_L(l;\lambda,\varepsilon_{dB},\omega_{db})=\begin{cases}\phi(\frac{10\log(l)-\varepsilon_{dB}}{\omega_{db}})-2\mathrm{T}(\frac{10\log(l)-\varepsilon_{dB}}{\omega_{db}},\lambda) & l>0\\ 0 \quad otherwise\end{cases} \tag{10}$$

where $\varphi(\mathrm{x})=\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}$, $\phi(\mathrm{x})=\int_{-\infty}^{x}\varphi(\zeta)\,\mathrm{d}\zeta$

and $\varepsilon_{dB}=\frac{\varepsilon}{\xi}$, $\omega_{dB}=\frac{\omega}{\xi}$, $\xi=\frac{\ln(10)}{10}$

C. Log Skew Normal Parameters Derivation

Let $\vec{L}$ be an n-dimensional log-normal vector with parameters μ and M. We define $B = M^{-1}$ as the inverse of covariance matrix. According to [5], λ_{opt} is defined as the following nonlinear equation:

$$\frac{\sum_{i=1}^{N} e^{2\mu_i} e^{\sigma_i^2}(e^{\sigma_i^2}-1)}{(\sum_{i=1}^{N} e^{\mu_i} e^{\sigma_i^2/2})^2} = e^{\frac{1+\lambda^2}{\sqrt{\sum_{1\le i,j\le N} B_{i,j}}}} \frac{\phi(2\frac{\lambda}{\sqrt{\sum_{1\le i,j\le N} B_{i,j}}})}{2\phi^2(\frac{\lambda}{\sqrt{\sum_{1\le i,j\le N} B_{i,j}}})} - 1 \qquad (11)$$

Such nonlinear equation can be solved using different mathematical utility (e.g. fsolve in Matlab). A starting solution guess λ_0 to (11) may be used in order to converge rapidly (only few iterations are needed):

$$\lambda_0 = \sqrt{\left[\operatorname{Max}_i\{B(i,i)\}^2 \sum_{1\le i,j\le N} B_{i,j})\right] - 1} \qquad (12)$$

Optimal location and scale parameters ε_{opt}, ω_{opt} are obtained according to λ_{opt} as:

$$\omega_{opt} = \sqrt{\frac{1+\lambda_{opt}^2}{\sum_{1\le i,j\le N} B_{i,j}}} \qquad (13)$$

$$\varepsilon_{opt} = \ln(\sum_{i=1}^{N} e^{\mu_i} e^{\sigma_i^2/2}) - \frac{\omega_{opt}^2}{2} - \ln(\phi(\frac{\lambda_{opt}}{\sqrt{\sum_{1\le i,j\le N} B_{i,j}}})) \qquad (14)$$

III. OUTAGE PROBABILITY COMPUTATION

We consider a homogeneous hexagonal network made of M rings around a central cell. Fig. 1 shows an example of such a network with the main parameters involved in the study: *R*, the cell range (1.5 km), *Rc*, the half-distance between BS. We focus on a mobile station (MS) u and its serving Base Station (BS), BS_i, surrounded by M interfering BS. For our system model, the SINR with N co-channel interferers at the receiver can be written in the following way:

$$SINR = \frac{S}{I + N_{th}} = \frac{P_i K r_{i,u}^{-\eta} . \mathrm{Y}_{i,u}}{\sum_{j=0, j\ne i}^{N} P_j K r_{j,u}^{-\eta} . \mathrm{Y}_{j,u} + N_{th}} \qquad (15)$$

The path-loss model is characterized by parameters K and $\eta > 2$. P_i is the transmission power of BS_i, the term $P_i K r_i^{-\eta}$ is the mean value of the received power at distance r_i from the base station BS_i. Shadowing effect is represented by lognormal random variable $Y_{i,u} = 10^{\frac{\zeta_{i,u}}{10}}$ where $\zeta_{i,u}$ is a normal RV, with zero mean and standard deviation σ, typically ranging from 3 to 12 dB.
N_{th} is the thermal noise power which can be neglected with respect to inter-cell interference. Furthermore, we assume that all base stations have identical transmitting powers. As we focus on a single User Equipment (UE), we may drop the indexes i,u and set:

$$r_{i,u} = r,\ Y_{i,u} = Y_0, \text{ and } \mathrm{Y}_{j,u} = \mathrm{Y}_j$$

The SINR perceived by the *UE* u can be written in the following way:

$$SINR = \frac{r^{-\eta} . \mathrm{Y}_0}{\sum_{j=1}^{N} r_j^{-\eta} . \mathrm{Y}_j} \qquad (16)$$

The outage probability is defined as the probability for the γ SINR to be lower than a threshold value δ:

$$P(\gamma < \delta) = P(\frac{1}{\delta} < \frac{\sum_{j=1}^{N} r_j^{-\eta} \mathrm{Y}_j}{r^{-\eta} . \mathrm{Y}_0}) \qquad (17)$$

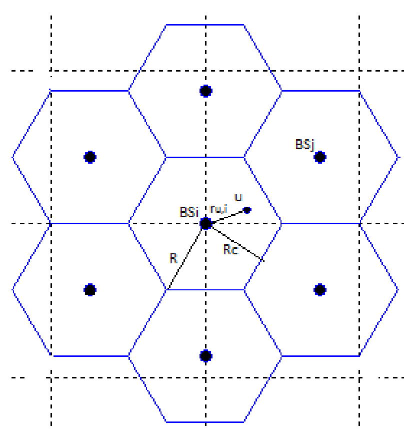

Figure 1. Hexagonal network and main parameters

Introducing the RV:

$$Z_f = \frac{\sum_{j=1}^{N} r_j^{-\eta}.\mathrm{Y}_j}{r^{-\eta}.\mathrm{Y}_0} \tag{18}$$

The outage probability is now expressed as:

$$P(\gamma < \delta) = P(\frac{1}{\delta} < Z_f) \tag{19}$$

Z_f is a location dependent factor. The numerator is a sum of lognormal RVs, which can be approximated by a log skew normal RV. For sake of simplicity, we define Λ as:

$$\Lambda = \sum_{j=1}^{N} r_j^{-\eta}.\mathrm{Y}_j = \sum_{j=1}^{N} \Lambda_j \tag{20}$$

So that Λ_j is a lognormal RV with mean $\mu_j = \log(r_j^{-\eta})$ and variance $\sigma_j = \xi.\sigma$ where $\xi = \frac{\ln(10)}{10}$.

$\vec{\Lambda} = (\Lambda_0, \Lambda_1 ... \Lambda_N)$ is a strictly positive random vector such that the vector $\vec{L} = (L_0, L_1 ... L_N)$ with $L_j = \log(\Lambda_j)$, $0 \le j \le N$ has an n-dimensional normal distribution with mean vector $\mu = (\mu_0, \mu_1 ... \mu_N)$ and covariance matrix M with $M(\mathrm{i,j}) = Cov(L_i, L_j)$, $0 \le i, j \le N$. $\vec{\Lambda}$ is called an n-dimensional log-normal vector with parameters $\vec{\mu}$ and M. Let R be the shadowing correlation matrix. We have:

$$R(i,j) = \begin{cases} 1 & if\ i = j \\ \rho & if\ i \ne j \end{cases} \tag{21}$$

The covariance matrix in the normal domain may be written as [10, Eq 11.71]:

$$M(i,j) = \ln\left\{\mathrm{R}(i,j).\sqrt{(\mathrm{e}^{\sigma_i^2}-1)(\mathrm{e}^{\sigma_j^2}-1)}+1\right\} \tag{22}$$

$Cov(\Lambda_i, \Lambda_j)$ may be expressed as:

$$Cov(\Lambda_i, \Lambda_j) = e^{\mu_i+\mu_j+\frac{1}{2}(\sigma_i^2+\sigma_j^2)}(\mathrm{e}^{M(i,j)}-1) \tag{23}$$

Let $B = M^{-1}$ the inverse of the covariance matrix. According to [5], Λ can be approximated by a Log Skew Normal distribution $LSN(\lambda_\Lambda, \omega_\Lambda, \varepsilon_\Lambda)$ where λ_Λ is defined as solution the nonlinear equation (11). Optimal location and scale parameters ε_Λ , ω_Λ are obtained according to λ_Λ as:

$$\omega_\Lambda = \sqrt{\frac{1+\lambda_\Lambda^2}{\sum_{1\le i,j\le N} B_{i,j}}} \tag{24}$$

$$\varepsilon_\Lambda = \ln(\sum_{i=1}^{N} e^{\mu_i} e^{\sigma_i^2/2}) - \frac{\omega_\Lambda^2}{2} - \ln(\phi(\frac{\lambda_\Lambda}{\sqrt{\sum_{1\le i,j\le N} B_{i,j}}})) \tag{25}$$

As $Z_f = \frac{\Lambda}{\Lambda_0}$, the correlation factor between $\ln(\Lambda)$ and $\ln(\Lambda_0)$ may be expressed as (see Appendix A):

$$r = \frac{\ln\left[Corr(\Lambda,\Lambda_0).\frac{\sqrt{e^{\sigma_0^2}-1}.\sqrt{2(e^{\omega_\Lambda^2}\varphi(2\beta_\Lambda\omega_\Lambda)-2\varphi^2(\beta_\Lambda\omega_\Lambda))}}{2\varphi(\beta_\Lambda\omega_\Lambda)}+1\right]}{\sigma_0\omega_\Lambda} \tag{26}$$

Where $\beta_\Lambda = \frac{\lambda_\Lambda}{\sqrt{1+\lambda_\Lambda^2}}$

We adopt the multivariate extension for skew normal distribution defined in [8]. As a quotient of two dependent log skew normal RVs, Z_f is a log skew normal distribution $LSN(\lambda_f, \omega_f, \varepsilon_f)$ (see Appendix B) where:

$$\lambda_f = \frac{\omega_\Lambda(\alpha_2 + r\alpha_1) - \sigma_0(\alpha_1 + r\alpha_2)}{\sqrt{\omega_f^2 + (1-r^2)(\alpha_1\omega_\Lambda + \alpha_2\sigma_0)^2}} \tag{27}$$

$$\omega_f = \sqrt{\omega_\Lambda^2 - 2r\omega_\Lambda\sigma_0 + \sigma_0^2} \tag{28}$$

$$\varepsilon_f = \varepsilon_\Lambda - \mu_0 \tag{29}$$

and

$$\vec{\alpha} = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} = \frac{\vec{\lambda}^T \Psi^{-1}\Delta^{-1}}{\sqrt{1+\vec{\lambda}^T\Psi^{-1}\vec{\lambda}}} \tag{30}$$

$$\Psi = \Delta^{-1}\Omega\Delta^{-1} - \vec{\lambda}\vec{\lambda}^T \tag{31}$$

$$\Delta = \begin{pmatrix}\sqrt{1-\delta_1^2} & 0 \\ 0 & \sqrt{1-\delta_2^2}\end{pmatrix} \tag{32}$$

$$\vec{\delta} = \begin{pmatrix} 0 \\ \dfrac{\lambda_\Lambda}{\sqrt{1+\lambda_\Lambda^2}} \end{pmatrix} \tag{33}$$

$$\vec{\lambda} = \begin{pmatrix} 0 \\ \lambda_\Lambda \end{pmatrix} \tag{34}$$

$$\Omega = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \tag{35}$$

Thus, the outage probability for a UE located at a distance r from its serving BS, can be written, using (20), as:

$$\begin{aligned} P(\gamma < \delta) &= 1 - P(\ln(Z_f) < \ln(\frac{1}{\delta})) \\ &= Q_f\left[\frac{\ln(\frac{1}{\delta}) - \varepsilon_f}{\omega_f}\right] \end{aligned} \tag{36}$$

Where $Q_f(x) = 1 - \phi(x) + 2T(x, \lambda_f)$

$\phi(x)$ is the standard normal CDF.

$T(x, \lambda)$ is the Owen's T function.

IV. VALIDATION

In this section, we propose to validate our formula for outage probability computation and compare it with simulation results. Fig. 2 show the outage probability for a UE located at cell edge (r=Rc) and inside the cell (r=Rc/2, r=Rc/4) with $\sigma = 10dB$, $\eta = 3.5$ and $\rho = 0.7$. We note that fluctuation at the tail of SINR distribution is due to Monte Carlo simulation, since we consider 10^7 samples at every turn. It is obvious that our approximations deliver same results as Monte Carlo simulations.

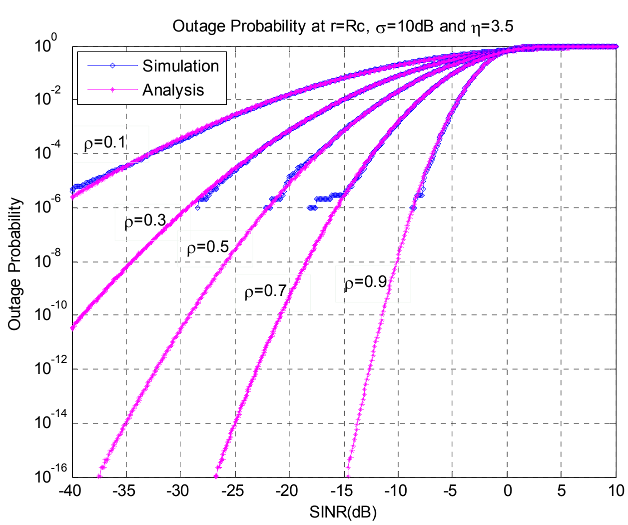

Figure 3. Outage probability at cell edge (r=Rc) for different correlation coefficients with $\sigma = 10dB$, $\eta = 3.5$

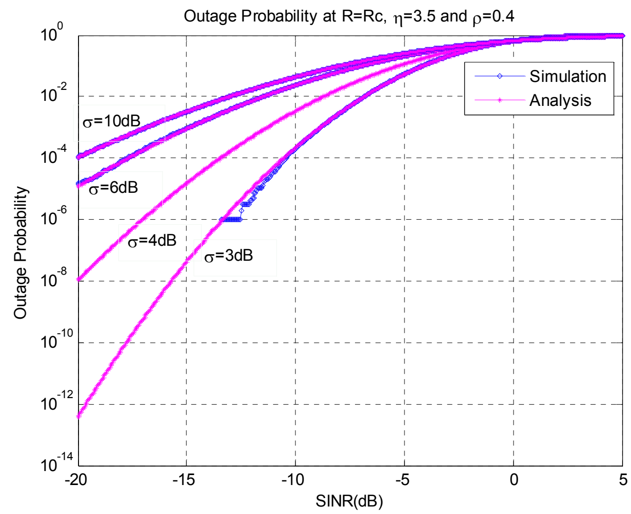

Figure 4. Outage probability at cell edge (r=Rc) for different standard deviation values with $\eta = 3.5$ and $\rho = 0.4$

To look into the effect of correlation coefficient on the accuracy of our formula, fig. 3 shows the outage probability for a UE located at cell edge with $\sigma = 10dB$ and $\eta = 3.5$ for different values of correlation coefficients ($\rho = 0.1 ... 0.9$). As we can see, proposed formula provides exact Monte Carlo simulation results independently of the value of the correlation factor.

Fig. 4 show the outage probability for a UE located at cell edge with $\rho = 0.4$ and $\eta = 3.5$ for different values of shadowing standard deviation ($\sigma = 3, 4, 6, 10\,dB$). One can see that the outage probability is accurately evaluated over the whole range of dB spreads.

Finally, we examine the effect of pathloss model on the accuracy of our approximation. Fig. 5 shows the outage probability for a UE located at cell edge with $\sigma = 10dB$ and $\rho = 0.9$ for different values of path loss exponent ($\eta = 2.5, 3.5, 4.5$). The accuracy of our approximations remains the same independently of pathloss model choice.

V. CONCLUSION

In this paper, we proposed a simple and highly accurate formula for outage probability computation in correlated lognormal shadowing environment. Our formula is based on the approximation of the sum of correlated lognormal RVs by a log skew normal distribution. LSN approximation provides identical results to Monte Carlo simulations independently of system parameters (correlation coefficients, standard deviation and pathloss exponent). The proposed formula can be used in performances evaluation of wireless networks since no more time-consuming Monte Carlo simulation is needed.

Appendix A: Computation of correlation factor between $\ln(\Lambda)$ and $\ln(\Lambda_0)$

As $Z_f = \frac{\Lambda}{\Lambda_0}$, to compute the correlation factor between Λ and Λ_0, we proceed in the following way :

$$
\begin{aligned}
Corr(\Lambda,\Lambda_0) &= \frac{\mathrm{Cov}(\Lambda,\Lambda_0)}{\sqrt{Var(\Lambda_0)}\sqrt{Var(\Lambda)}} \\
&= \frac{\sum_{j=1}^{N} Cov(\Lambda_j,\Lambda_0)}{\sqrt{Var(\Lambda_0)}\sqrt{Var(\Lambda)}} \\
&= \frac{\sum_{j=1}^{N} e^{\mu_0+\mu_j+\frac{1}{2}(\sigma_0^2+\sigma_j^2)}(\mathrm{e}^{M(0,j)}-1)}{\sqrt{Var(\Lambda_0)}\sqrt{Var(\Lambda)}}
\end{aligned}
$$

Using the moment generating function of the bivariate skew normal distribution, defined in [8], we may write:

$$
\begin{aligned}
COV(\Lambda\Lambda_0) &= E[\Lambda\Lambda_0] - E[\Lambda]E[\Lambda] \\
&= 2\varphi(\beta\omega)\left(\exp(r\omega\sigma_0)-1\right)\exp\left\{\varepsilon+\mu_0+\frac{1}{2}(\omega^2+\sigma_0^2)\right\}
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
COV(\Lambda\Lambda_0) &= Corr(\Lambda,\Lambda_0).\sqrt{Var(\Lambda_0)}.\sqrt{Var(\Lambda)} \\
&= Corr(\Lambda,\Lambda_0)\sqrt{e^{\sigma_0^2}-1}.\sqrt{2(e^{\omega^2}\phi(2\beta\omega)-2\phi^2(\beta\omega))} \\
&\quad .\exp\left\{\varepsilon+\mu_0+\frac{1}{2}(\omega^2+\sigma_0^2)\right\}
\end{aligned}
$$

The correlation factor between $\ln(\Lambda)$ and $\ln(\Lambda_0)$ may be expressed as:

$$
r = \frac{\ln\left[Corr(\Lambda,\Lambda_0).\frac{\sqrt{e^{\sigma_0^2}-1}.\sqrt{2(e^{\omega^2}\varphi(2\beta\omega)-2\varphi^2(\beta\omega))}}{2\varphi(\beta\omega)}+1\right]}{\sigma_0\omega}
$$

Appendix B: Difference of two dependent skew normal RVs

Given two skew normal dependent random variables $(X,Y) \sim \mathrm{SN}_2(\Psi,\vec{\alpha})$ as defined in [8], i.e. the joint distribution of X and Y is giving by:

$$
\begin{aligned}
f(x,y) =& \frac{2}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\} \\
&\Phi\left\{\alpha_1\frac{(x-\mu_1)}{\sigma_1}+\alpha_2\frac{(y-\mu_2)}{\sigma_2}\right\}
\end{aligned}
$$

Then the PDF of the difference $Z = Y - X$ is given by:

$$
\begin{aligned}
\varphi_z(\mathrm{z}) =& \int_{-\infty}^{+\infty} f(x,z+x)\mathrm{d}x \\
=& \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x-\mu_1)(z+x-\mu_2)}{\sigma_1\sigma_2}+\frac{(z+x-\mu_2)^2}{\sigma_2^2}\right]\right\} \\
&\Phi\left\{\alpha_1\frac{(x-\mu_1)}{\sigma_1}+\alpha_2\frac{(z+x-\mu_2)}{\sigma_2}\right\}\mathrm{d}x
\end{aligned}
$$

Let $t = \frac{x-\mu_1}{\sigma_1}$, for more convenience, we define $\mu = \mu_2 - \mu_1$, so that:

$$
\begin{aligned}
\varphi_z(\mathrm{z}) =& \frac{1}{\pi\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[t^2-2\rho t\left(\frac{z-\mu+\sigma_1 t}{\sigma_2}\right)+\left(\frac{z-\mu+\sigma_1 t}{\sigma_2}\right)^2\right]\right\} \\
&\Phi\left\{\alpha_1 t+\alpha_2\left(\frac{z-\mu+\sigma_1 t}{\sigma_2}\right)\right\}\mathrm{d}t
\end{aligned}
$$

Let $x = \left[\alpha_1+\alpha_2\frac{\sigma_1}{\sigma_2}\right]\mathrm{t}+\alpha_2\left[\frac{z-\mu}{\sigma_2}\right]$, after some algebra, we may write:

$$
\begin{aligned}
\varphi_z(\mathrm{z}) =& \frac{1}{\pi(\alpha_1\sigma_2+\alpha_2\sigma_1)\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}\left[\nu x-\frac{\beta}{\nu}(\mathrm{z}-\mu)\right]^2\right\}\Phi(x)\mathrm{d}x \\
&-\frac{(\mathrm{z}-\mu)^2}{2\nu^2}\Phi(x)\mathrm{d}x
\end{aligned}
$$

where

$$
\nu = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}
$$

$$
\beta = \sigma_2(\alpha_2+\rho\alpha_1) - \sigma_1(\alpha_1+\rho\alpha_2)
$$

Given the fact that $\Phi(x) = \frac{1}{2}(1+erf(\frac{x}{\sqrt{2}}))$, we may write:

$$
\varphi_z(\mathrm{z}) = (A_1+A_2)\exp\left\{-\frac{(\mathrm{z}-\mu)^2}{2\nu^2}\right\}
$$

where :

$$
A_1 = \frac{1}{2\pi(\alpha_1\sigma_2+\alpha_2\sigma_1)\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}(\nu x-\frac{\beta}{\nu}(\mathrm{z}-\mu))^2\right\}\mathrm{d}x
$$

Let $y = \nu x - \frac{\beta}{\nu}(z-\mu)$, we may write :

$$A_1 = \frac{1}{2\pi\nu\left(\alpha_1\sigma_2+\alpha_2\sigma_1\right)\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}\,y^2\right\}\mathrm{d}y;$$

$$= \frac{1}{2\pi\nu}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2}t^2\right\}\mathrm{d}t; \quad t = \frac{1}{\sqrt{(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)}}\,y$$

$$= \frac{1}{\sqrt{2\pi}\nu}$$

and

$$A_2 = \frac{1}{2\pi\left(\alpha_1\sigma_2+\alpha_2\sigma_1\right)\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}\,(\nu x-\frac{\beta}{\nu}(z-\mu))^2\right\}erf(\frac{x}{\sqrt{2}})\mathrm{d}x$$

Let $(y = \frac{x}{\sqrt{2}})$, we may write :

$$A_2 = \frac{1}{\sqrt{2}\pi\left(\alpha_1\sigma_2+\alpha_2\sigma_1\right)\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}\,(\sqrt{2}\nu y-\frac{\beta}{\nu}(z-\mu))^2\right\}erf(y)\mathrm{d}y$$

Using [11, Equation 13]:

$$\int_{-\infty}^{+\infty}\exp\left\{-(ax+b)^2\right\}erf(x)\,\mathrm{d}x = -\frac{\sqrt{\pi}}{a}erf(\frac{b}{\sqrt{a^2+1}})$$

We may write:

$$A_2 = \frac{-1}{\sqrt{2\pi}\nu}erf(\frac{-\beta(z-\mu)}{\nu\sqrt{2}\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})$$

$$= \frac{-1}{\sqrt{2\pi}\nu}\left[2\Phi(\frac{-\beta(z-\mu)}{\nu\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})-1\right]$$

So that:

$$\varphi_z(z) = \frac{\sqrt{2}}{\sqrt{\pi}\nu}\left[1-\Phi(\frac{-\beta(z-\mu)}{\nu\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})\right]\exp\left\{-\frac{(z-\mu)^2}{2\nu^2}\right\}$$

Let $F_z(z)$ denotes the cumulative distribution function of the difference $Z = Y - X$, we have:

$$F_z(z) = \int_{-\infty}^{z}\varphi_z(x)dx$$

$$= \int_{-\infty}^{z}\frac{\sqrt{2}}{\sqrt{\pi}\nu}\exp\left\{-\frac{(x-\mu)^2}{2\nu^2}\right\}dx$$

$$-\int_{-\infty}^{z}\frac{\sqrt{2}}{\sqrt{\pi}\nu}\Phi(\frac{-\beta(x-\mu)}{\nu\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})\exp\left\{-\frac{(x-\mu)^2}{2\nu^2}\right\}dx$$

Let $y = \frac{(x-\mu)}{\nu}$, we may write :

$$F_z(z) = 2\int_{-\infty}^{\frac{(z-\mu)}{\nu}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}y^2\right\}dy$$

$$-\int_{-\infty}^{\frac{(z-\mu)}{\nu}}\frac{2}{\sqrt{2\pi}}\Phi(\frac{-\beta y}{\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})\exp\left\{-\frac{1}{2}y^2\right\}dy$$

$$= 2\Phi(\frac{z-\mu}{\nu}) - S_k(\frac{z-\mu}{\nu},\frac{-\beta}{\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})$$

$$= 2\Phi(\frac{z-\mu}{\nu}) - \Phi(\frac{z-\mu}{\nu}) + 2\,T(\frac{z-\mu}{\nu},\frac{-\beta}{\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})$$

where $T(x,\lambda)$ is the Owen's T function expressed as:

$$T(x,\lambda) = \frac{1}{2\pi}\int_0^{\lambda}\frac{\exp\left\{-\frac{1}{2}x^2(1+t^2)\right\}}{(1+t^2)}dt$$

We have then:

$$F_z(z) = \Phi(\frac{z-\mu}{\nu}) - 2\,T(\frac{z-\mu}{\nu},\frac{\beta}{\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})$$

$$= SN(\frac{z-\mu}{\nu},\frac{\beta}{\sqrt{\nu^2+(1-\rho^2)(\alpha_1\sigma_2+\alpha_2\sigma_1)^2}})$$

where

$$\nu = \sqrt{\sigma_1{}^2 - 2\rho\sigma_1\sigma_2 + \sigma_2{}^2}$$
$$\beta = \sigma_2(\alpha_2+\rho\alpha_1) - \sigma_1(\alpha_1+\rho\alpha_2)$$
$$\mu = \mu_2 - \mu_1$$

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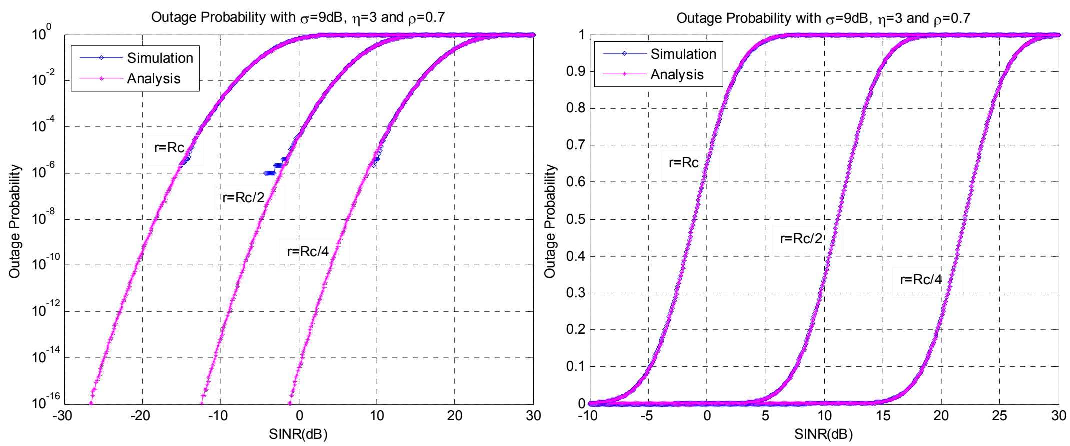

Figure 2. Outage probability at cell edge (r=Rc) and inside the cell (r=Rc/2, r=Rc/4) with $\sigma = 10dB$, $\eta = 3.5$ and $\rho = 0.7$

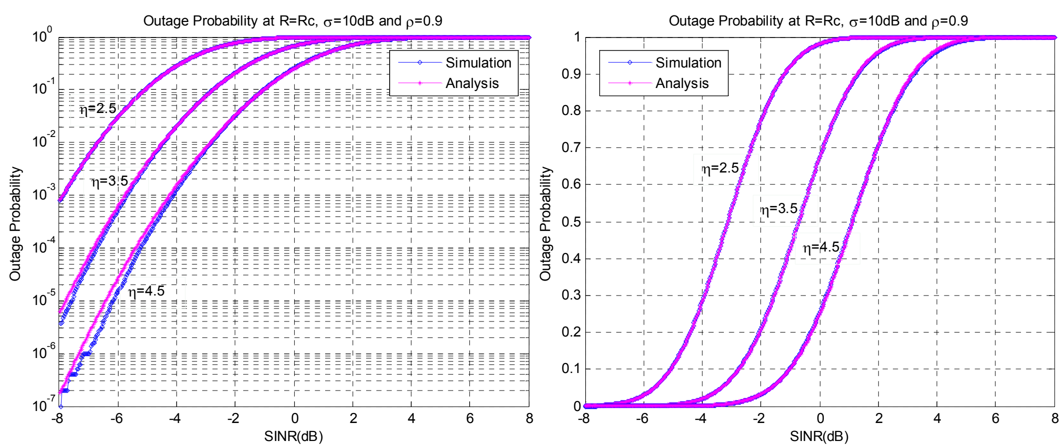

Figure 5. Outage probability at cell edge (r=Rc) for different path loss exponent values with $\sigma = 10dB$ and $\rho = 0.9$