SIR Distribution and Scheduling Gain of Normalized SNR Scheduling in Poisson Networks

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***Abstract*—In most stochastic geometry analyses of cellular networks, a receiving user is selected randomly. Thus, the network performance is equivalent to that obtained with the use of a round-robin scheduler; in other words, channel-aware user scheduling is not applied. The first objective of this study is to clarify the type of channel-aware user scheduling that can be utilized with stochastic geometry. In both Poisson bipolar networks and Poisson cellular networks, the signal-to-interference power ratio (SIR) distribution, average data rate, and scheduling gain of a normalized SNR scheduler are derived. The scheduler selects the user with the largest SNR normalized by the short-term average SNR, and it is equivalent to the well-known proportional fair scheduler when the data rate is proportional to the SNR. The second objective is to discuss the achievable scheduling gain with stochastic geometry when compared with round-robin scheduling. The value of the scheduling gain is examined using numerical integration results.**

I. Introduction

Stochastic geometry has been applied to various kinds of wireless networks including cellular networks for performance analysis [1]–[4]. In our previous work [5], the complementary cumulative distribution function (ccdf) of the signal-to-interference-plus-noise power ratio (SINR), $\mathbb{P}(SINR > \theta)$, average data rate, and scheduling gain in downlink Poisson cellular networks with the *normalized SNR scheduler* were derived. The purpose of the present paper is to provide additional information and supplement to the previous work [5].

The spatial distribution of SINR in Poisson cellular networks was first derived in [1], and has been analyzed in a wide variety of scenarios [3], [4]. However, in terms of stochastic geometry analyses of Poisson cellular networks, channel-aware user scheduling has not been applied.

In a single-cell environment, the normalized SNR scheduler and proportional fair (PF) scheduler [6] were analyzed in [7], [8], respectively. In a multi-cell environment, the normalized SNR scheduler were analyzed assuming that interference is independent of time and the locations of users in [9] and assuming hexagonal cell arrangements [10]. We note that these studies [9], [10] evaluated average throughput of the network, not the spatial distribution of SINR.

The contributions of the paper are:

- We derive the signal-to-interference power ratio (SIR) ccdf[1] in multiuser extended Poisson bipolar networks. Further, we derive the SIR ccdf in Poisson cellular networks using hypergeometric functions.
- We discuss the *scheduling gain* [11] in detail, which is defined as the ratio of the average data rate of the normalized SNR scheduler to that of the round-robin scheduler.

This paper is organized as follows: In Section II, we review previous results on user scheduling. In Sections III and IV, the SIR distribution and average data rate of both multiuser extended Poisson bipolar networks and Poisson cellular networks. Section V concludes the paper.

Notation: $\mathbb{E}[\cdot]$ denotes the expectation operator, $f_X(\cdot)$ the probability density function (pdf) of a continuous random variable X or the probability mass function (pmf) of a discrete random variable X, $F_X(\cdot)$ the cumulative distribution function (cdf) of X, $\mathcal{L}_X(\cdot)$ the Laplace transform of the pdf of X, $\Gamma(\cdot)$ the gamma function, $\mathrm{B}(\cdot,\cdot)$ the beta function, $E_1(\cdot)$ the exponential integral, and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ the hypergeometric function.

II. User Scheduling

A. System Model

We consider a situation where m users are associated with a base station (BS). In each time slot, the BS transmits to one user with unit power based on the following scheduler.

B. Round-Robin Scheduler

The round-robin scheduler selects one user in rotation independent of the channel and interference. Thus, each user receives a fraction $1/m$ of the time slots.

C. Normalized SNR Scheduler

We consider a situation where users experience quasi-static Rayleigh fading, i.e., the channel gain is constant over a time slot. The instantaneous SNR of the ith user at distance r_i is given as

$$SNR_i := h_i r_i^{-\alpha}/\sigma^2, \tag{1}$$

where h_i represents the exponentially distributed fading gain with unit mean, i.e., $h_i \sim \exp(1)$, $\alpha > 2$ the path loss exponent, and σ^2 the noise power.

[1]The SIR ccdf is also referred to as the success probability or coverage probability.

For any time slot, the BS is assumed to have knowledge of the vector of the instantaneous SNR $(SNR_j)_{1\le j\le m}$ according to perfect channel estimation at the beginning of each time slot, as described in [6], [7], [9], [12]. The scheduler selects the user with the largest instantaneous SNR normalized by the *short-term* average SNR. The short-term average SNR is the average value of the instantaneous SNR over a period when variation in the distance between the user and its associated BS is negligible, as presented in [6], [7]. Here, we note that the scheduler is not aware of the interference, but the performance measure is the SIR.

The short-term average SNR is given by the conditional expectation of SNR_i given r_i. Because r_i and h_i are independent and $\mathbb{E}[h_i] = 1$, we have

$$\mathbb{E}[SNR_i \,|\, r_i] = r_i^{-\alpha}/\sigma^2. \tag{2}$$

With the use of (1) and (2), the instantaneous SNR normalized by the short-term average SNR can be given as

$$\frac{SNR_i}{\mathbb{E}[SNR_i \,|\, r_i]} = h_i, \tag{3}$$

i.e., the fading gain of user i. Here, we note that if the data rate is proportional to the SNR, the normalized SNR scheduler is equivalent to the PF scheduler [8].

User i is selected when

$$h_i = \max_{j=1,\dots,m} h_j. \tag{4}$$

For the sake of notational simplicity, we define

$$h_m^\star := \max_{j=1,\dots,m} h_j. \tag{5}$$

Because $(h_j)_{1\le j\le m}$ are statistically identical, we have

$$\mathbb{P}(h_i = h_m^\star) = 1/m, \ \forall i, \tag{6}$$

i.e., each user receives a fraction $1/m$ of the time slots, as in the case of round-robin scheduler.

We derive the cdf of the channel gain when the user i is scheduled,

$$\mathbb{P}(h_i \le x \,|\, h_i = h_m^\star) = \mathbb{P}(h_m^\star \le x) = F_{h_m^\star}(x). \tag{7}$$

Because $(h_j)_{1\le j\le m}$ are independent and identically distributed (i.i.d.) exponential random variables with unit mean, the distribution of $h_m^\star$ is the largest-order statistic [13] and derived as in the case of the selection combiner output [14].

$$\begin{aligned} F_{h_m^\star}(x) &= \big(F_{h_j}(x)\big)^m = \big(1 - \mathrm{e}^{-x}\big)^m \\ &\overset{\text{(a)}}{=} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \mathrm{e}^{-kx}, \end{aligned} \tag{8}$$

where (a) follows from the binomial theorem.

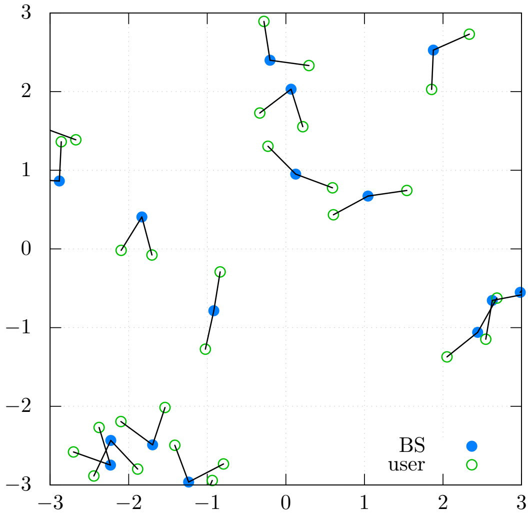

Fig. 1: Realization of a multiuser extended Poisson bipolar network for $\lambda = 1/2$, $r = 1/2$, and $m = 2$.

D. Scheduling Gain

To be compared with scheduling gains of Poisson networks, we introduce a simple evaluation formula for scheduling gain under Rayleigh fading channel [11], which is achieved when the data rate is proportional to the SNR:

$$\begin{aligned} \frac{\mathbb{E}[h_i \,|\, h_i = h_m^\star]}{\mathbb{E}[h_i]} &= \int_0^\infty \big(1 - F_{h_m^\star}(x)\big)\,\mathrm{d}x \\ &= \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \frac{1}{k} = \sum_{k=1}^{m} \frac{1}{k}. \end{aligned} \tag{9}$$

As to be presented in this paper, scheduling gains in Poisson networks are also written in forms of an alternating binomial sum as (9) but cannot be written as a sum without alternating signs and binomial coefficients.

III. Poisson Bipolar Networks

A. System Model and Previous Results

In a *Poisson bipolar network* [15, Def. 5.8], the transmitters form a Poisson point process (PPP) Φ with intensity λ, and all transmitters have a receiver at an identical distance r with a random orientation. For the sake of consistency with other sections in this paper, the transmitters and receivers are referred to as BSs and users, respectively. We define $C := \lambda\pi r^2\,\Gamma(1 + 2/\alpha)\,\Gamma(1 - 2/\alpha)$, with I denoting the aggregated interference at the intended user.

The SIR ccdf is given as[2]

$$\begin{aligned} \mathbb{P}(SIR > \theta) &= \mathbb{E}\big[\mathbb{P}(h > \theta r^\alpha I \,|\, I)\big] = \mathbb{E}\big[1 - F_h(\theta r^\alpha I)\big] \\ &= \mathbb{E}\big[\mathrm{e}^{-\theta r^\alpha I}\big] = \mathcal{L}_I(\theta r^\alpha) = \mathrm{e}^{-C\theta^{2/\alpha}}. \end{aligned} \tag{10}$$

We extend the Poisson bipolar network to accommodate multiple users, i.e., all BSs have an identical number m of

[2]Strictly speaking, in this case, $\mathbb{P}$ should be written as $\mathbb{P}^{!t}$, which represents the reduced Palm measure given that there is a BS at a prescribed location and the SIR is measured at the intended user.

users at an identical distance r with random orientations, as shown in Fig. 1. Each BS selects one user to transmit according to the normalized SNR scheduler. The purpose of this model is to provide a simpler model and expressions when compared with the Poisson cellular networks presented in Section IV.

B. SIR of Scheduled User

Proposition 1. The SIR ccdf when user i is scheduled is given as

$$\mathbb{P}\Big(SIR_i > \theta \,\Big|\, h_i = \max_{j=1,\dots,m} h_j\Big) = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \mathrm{e}^{-C(k\theta)^{2/\alpha}}. \quad (11)$$

We note that when $m = 1$, (11) is equivalent to the SIR ccdf without channel-aware scheduling (10).

Proof.

$$\begin{aligned}
\mathbb{P}\Big(SIR_i > \theta \,\Big|\, h_i = \max_{j=1,\dots,m} h_j\Big) &= \mathbb{E}\big[\mathbb{P}(h_i > \theta r^\alpha I \,|\, I, h_i = h_m^\star)\big] \\
&= \mathbb{E}\big[\mathbb{P}(h_m^\star > \theta r^\alpha I \,|\, I)\big] = \mathbb{E}\big[1 - F_{h_m^\star}(\theta r^\alpha I)\big] \\
&= \mathbb{E}\Bigg[1 - \sum_{k=0}^{m} \binom{m}{k} (-1)^k \mathrm{e}^{-k\theta r^\alpha I}\Bigg] \\
&= \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1}\, \mathbb{E}\big[\mathrm{e}^{-k\theta r^\alpha I}\big] \\
&\overset{(a)}{=} \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \mathrm{e}^{-C(k\theta)^{2/\alpha}},
\end{aligned}$$

where (a) is obtained by replacing the θ in (10) by $k\theta$. We note that because the cdf of the channel gain of the scheduled user (8) is expressed in the form of a sum of exponential functions, it is possible to obtain term-wise expectation and apply the same derivation as in (10). □

Hereafter, for the sake of notational simplification, let the SIR of the scheduled user among m users be denoted by $SIR_m^\star$, e.g., $\mathbb{P}(SIR_m^\star > \theta) = \mathbb{P}(SIR_i > \theta \,|\, h_i = h_m^\star)$.

Fig. 2 shows the SIR ccdf (11). Along with the number of associated users of each BS, m, the SIR increases due to multiuser diversity gain.

C. Average Data Rate and Scheduling Gain

Proposition 2. The average data rate can be expressed as

$$\begin{aligned}
\mathbb{E}\big[\ln(1 + SIR_i) \,\big|\, h_i = h_m^\star\big] &= \mathbb{E}\big[\ln(1 + SIR_m^\star)\big] \\
&= \int_0^\infty \frac{\mathbb{P}(SIR_m^\star > \theta)}{1+\theta}\,\mathrm{d}\theta \\
&= \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \int_0^\infty \frac{\mathrm{e}^{-C(k\theta)^{2/\alpha}}}{1+\theta}\,\mathrm{d}\theta. \quad (12)
\end{aligned}$$

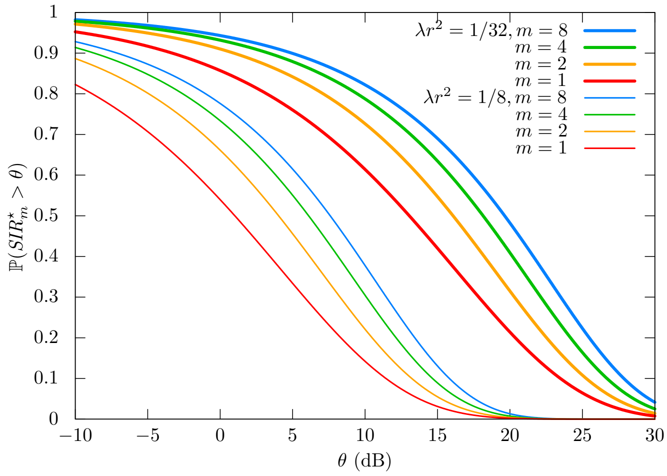

Fig. 2: SIR ccdf in multiuser extended Poisson bipolar networks for $\alpha = 4$.

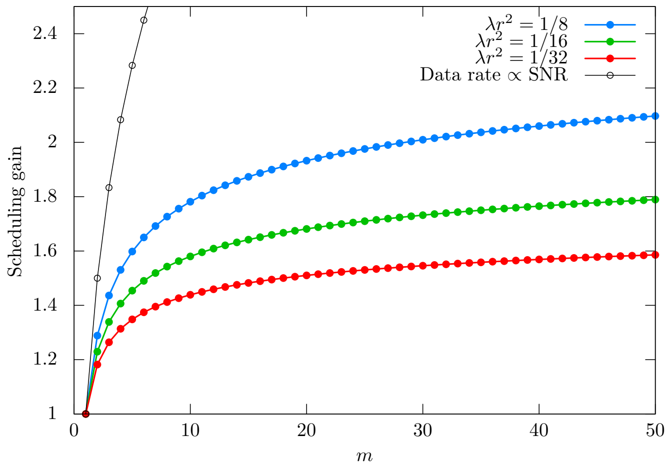

Fig. 3: Scheduling gain in multiuser extended Poisson bipolar networks for $\alpha = 4$.

Remark 1. For the convenience of numerical evaluation, (12) can be rewritten when $\alpha = 4$,

$$\mathbb{E}\big[\ln(1 + SIR_m^\star)\big] = 2 \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1}\, \mathrm{Re}\big(\mathrm{e}^{\mathrm{j}C\sqrt{k}}\, E_1(\mathrm{j}C\sqrt{k})\big), \quad (13)$$

where $C = \lambda \pi^2 r^2/2$.

Fig. 3 illustrates the scheduling gain $\mathbb{E}\big[\ln(1 + SIR_m^\star)\big]/\mathbb{E}\big[\ln(1 + SIR)\big]$, wherein it is confirmed that the scheduling gain increases with increase in the number of users m.

IV. Poisson Cellular Networks

In the previous section, we considered Poisson-distributed BSs with an identical number of users at identical distances. We now consider *Poisson cellular networks* [1], where the distance from the associated BS and the number of associated users per BS are treated as random variables.

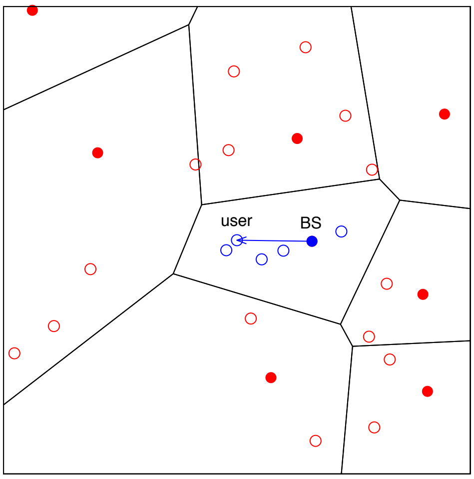

Fig. 4: Realization of a Poisson cellular network.

A. System Model and Previous Results

As shown in Fig. 4, the locations of the BSs and users are assumed to form two independent PPPs Φ_b with intensity λ_b and Φ_u with intensity λ_u, respectively, as in [2], [16]. Each user is associated with the nearest BS [2], i.e., the cell of each BS comprises a Voronoi tessellation. The typical user is assumed to be located at origin o, and with the location of the associated BS being denoted by $b_o := \arg\min_{x\in\Phi_\mathrm{b}}\|x\|$. Let R denote the distance between the typical user o and the associated BS b_o, i.e., $R := \|b_o\|$. According to [1], [15], the pdf of R can be written as

$$f_R(r) = 2\lambda_\mathrm{b}\pi r \mathrm{e}^{-\lambda_\mathrm{b}\pi r^2}. \tag{14}$$

From Slivnyak's theorem [15], the locations of the other users follow the reduced Palm distribution with Φ_u. Let the number of users in cell b_o except for the typical user o be denoted by N. We note here that N is a random variable whereas m in Section III is a constant. According to [16, Lemma 1], the pmf of N is given by

$$f_N(n) = \frac{(\lambda_\mathrm{u}/c\lambda_\mathrm{b})^n}{c\,\mathrm{B}(n+1,c)\,(\lambda_\mathrm{u}/c\lambda_\mathrm{b}+1)^{n+c+1}},\quad c = 3.5. \tag{15}$$

For the sake of simplicity, we consider a scenario where all BSs continuously transmit signals independently of the number of associated users, i.e., interference is generated from all the other BSs. The scenario where BSs with no user to serve do not transmit any signals has been discussed in [5].

Let the interference at the typical user o from BSs $\Phi_\mathrm{b}\setminus\{b_o\}$ be denoted by I_r. According to [1, Theorem 2], the SIR ccdf is given as[3]

$$\begin{aligned}\mathbb{P}(\mathit{SIR} > \theta) &= \mathbb{E}\big[\mathbb{P}(\mathit{SIR} > \theta \,|\, R = r)\big]\\ &= \int_0^\infty \mathcal{L}_{I_r}(\theta r^\alpha \,|\, r)\, f_R(r)\,\mathrm{d}r\\ &= \frac{1}{1 + \frac{2\theta}{\alpha-2}\,{}_2F_1(1, 1-2/\alpha; 2-2/\alpha; -\theta)},\end{aligned} \tag{16}$$

[3]Strictly speaking, $\mathbb{P}$ should be written as $\mathbb{P}^o$, which represents the Palm measure given that there is a user at o and the SIR is measured at the user.

$$\begin{aligned}&\mathcal{L}_{I_r}(\theta r^\alpha \,|\, r)\\ &\quad= \exp\left[-\frac{2\lambda_\mathrm{b}\pi r^2\theta}{\alpha-2}\,{}_2F_1\left(1, 1-\frac{2}{\alpha}; 2-\frac{2}{\alpha}; -\theta\right)\right].\end{aligned} \tag{17}$$

When $\alpha = 4$, $\mathbb{P}(\mathit{SIR} > \theta) = \frac{1}{1+\sqrt{\theta}\arctan\sqrt{\theta}}$ and $\mathbb{P}(\mathit{SIR} > \theta \,|\, r) = \mathcal{L}_{I_r}(\theta r^\alpha \,|\, r) = \exp(-\lambda_\mathrm{b}\pi r^2\sqrt{\theta}\arctan\sqrt{\theta})$. Hereafter, we consider expressions corresponding to $\alpha = 4$ for notational simplicity. It is noteworthy that expressions for $\alpha \neq 4$ can be obtained by using (16) and (17).

According to [17] and [1, Theorem 3], the average data rate defined as $\mathbb{E}\big[\ln(1+\mathit{SIR})\big]$ is given by

$$\begin{aligned}\mathbb{E}\big[\ln(1+\mathit{SIR})\big] &= \int_0^\infty \frac{\mathbb{P}(\mathit{SIR} > \theta)}{1+\theta}\,\mathrm{d}\theta\\ &\approx 1.489\,\mathrm{nat/s/Hz}.\end{aligned} \tag{18}$$

B. SIR of Scheduled User

Lemma 1 ([5, Lemma 1]). Given that the typical user o with fading gain h_o is scheduled, the ccdf of SIR conditioning on $R = r$ and $N = n$, $\mathbb{P}(\mathit{SIR} > \theta \,|\, r, n, h_o = \max_{j=o,1,\dots,n} h_j)$, is given by the sum of the Laplace transforms of the pdf of interference.

Proof.
$$\begin{aligned}&\mathbb{P}\Big(\mathit{SIR} > \theta \,\Big|\, r, n, h_o = \max_{j=o,1,\dots,n} h_j\Big)\\ &= \mathbb{P}(h_o > \theta r^\alpha I_r \,|\, r, n, h_o = h^\star_{n+1})\\ &= \mathbb{P}(h^\star_{n+1} > \theta r^\alpha I_r \,|\, r, n)\\ &= \mathbb{E}\big[\mathbb{P}(h^\star_{n+1} > \theta r^\alpha I_r \,|\, r, n, I_r) \,\big|\, r, n\big]\\ &= \mathbb{E}\big[1 - F_{h^\star_{n+1}}(\theta r^\alpha I_r) \,\big|\, r, n\big]\\ &= \mathbb{E}\left[1 - \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k \mathrm{e}^{-k\theta r^\alpha I_r} \,\middle|\, r, n\right]\\ &= 1 - \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k\,\mathbb{E}\big[\mathrm{e}^{-k\theta r^\alpha I_r} \,\big|\, r, k\big]\\ &= \sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k+1}\,\mathcal{L}_{I_r}(k\theta r^\alpha \,|\, r, k). \quad \square\end{aligned}$$

Hereafter, for the sake of notational simplification, we denote SIR as $\mathit{SIR}^\star$ when the typical user is scheduled, e.g., $\mathbb{P}(\mathit{SIR} > \theta \,|\, r, n, h_o = \max_{j=o,1,\dots,n} h_j) = \mathbb{P}(\mathit{SIR}^\star > \theta \,|\, r, n)$.

Proposition 3 ([5, Prop. 1]). Given that the typical user o is scheduled, the SIR ccdf can be approximated by

$$\begin{aligned}\mathbb{P}(\mathit{SIR}^\star > \theta) \approx \sum_{n=0}^{\infty} &\frac{(\lambda_\mathrm{u}/c\lambda_\mathrm{b})^n}{c\,\mathrm{B}(n+1,c)(\lambda_\mathrm{u}/c\lambda_\mathrm{b}+1)^{n+c+1}}\\ &\times \sum_{k=1}^{n+1}\frac{\binom{n+1}{k}(-1)^{k+1}}{1+\sqrt{k\theta}\arctan\sqrt{k\theta}}.\end{aligned} \tag{19}$$

That is, the distribution of $\mathit{SIR}^\star$ depends only on θ and $\lambda_\mathrm{u}/\lambda_\mathrm{b}$, while the distribution of SIR (16) depends only on θ but not on λ_b.

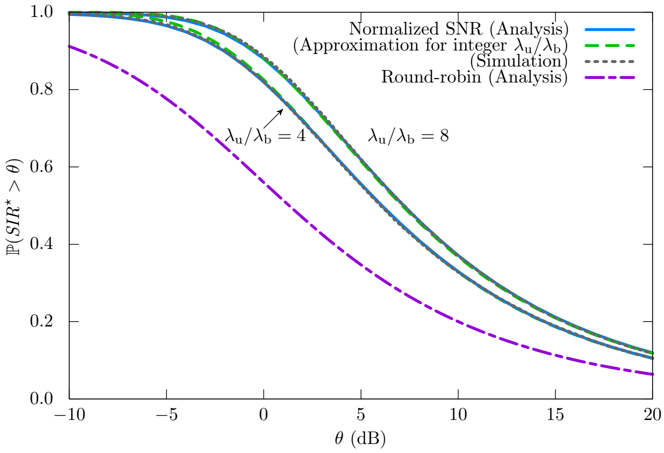

Fig. 5: SIR ccdf in Poisson cellular networks for $\alpha = 4$.

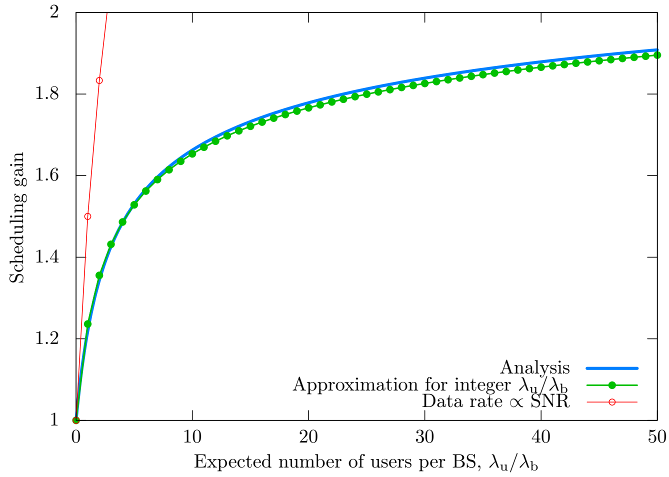

Fig. 6: Scheduling gain in Poisson cellular networks for $\alpha = 4$.

Proof.
$$\begin{aligned}\mathbb{P}(SIR^\star > \theta) &= \mathbb{E}\big[\mathbb{P}(SIR^\star > \theta \,|\, r, n)\big] \\ &\overset{(a)}{\approx} \sum_{n=0}^{\infty} \int_0^\infty \mathbb{P}(SIR^\star > \theta \,|\, r, n)\, f_R(r)\, f_N(n)\, \mathrm{d}r \\ &= \sum_{n=0}^{\infty} f_N(n) \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k+1} \\ &\qquad \times \int_0^\infty \mathcal{L}_{I_r}(k\theta r^\alpha \,|\, r, k)\, f_R(r)\, \mathrm{d}r \\ &\overset{(b)}{=} \sum_{n=0}^{\infty} f_N(n) \sum_{k=1}^{n+1} \frac{\binom{n+1}{k}(-1)^{k+1}}{1 + \sqrt{k\theta}\arctan\sqrt{k\theta}},\end{aligned}$$
where the joint probability density and mass function of R and N is approximated by the product of its marginals $f_R(r)$ and $f_N(n)$[4] though R and N are inherently dependent on each other, (a) follows from the approximation. We subsequently confirm the accuracy of the approximation by numerical evaluation. Further, (b) is obtained by means of the same derivation as (16). □

Remark 2. For numerical evaluation, it is convenient if there is no infinite sum in (19), which is due to the expectation with respect to the pmf $f_N(n)$. To avoid the calculation of this infinite sum, for integer points of λ_u/λ_b, we roughly approximate $f_N(n)$ as
$$\tilde{f}_N(n) = \begin{cases} 1, & n = \lambda_u/\lambda_b; \\ 0, & n \neq \lambda_u/\lambda_b. \end{cases} \tag{20}$$
We note that when λ_u/λ_b is an integer, both $\lambda_u/\lambda_b - 1$ and λ_u/λ_b represent the modes of N. Here, we use the latter mode λ_u/λ_b and set $\tilde{f}_N(\lambda_u/\lambda_b) = 1$. Using $\tilde{f}_N(n)$, the SIR ccdf is further approximated to
$$\mathbb{P}(SIR^\star > \theta) \approx \sum_{k=1}^{\lambda_u/\lambda_b+1} \frac{\binom{\lambda_u/\lambda_b+1}{k}(-1)^{k+1}}{1 + \sqrt{k\theta}\arctan\sqrt{k\theta}}. \tag{21}$$

Fig. 5 shows numerical results of the SIR ccdf (19) and its approximation for integer points of λ_u/λ_b (21) as obtained via numerical integration, and Monte Carlo simulation results. Since their characteristics are almost identical, we can conclude that the approximations made thus far are sufficient for evaluation.

In Fig. 5, for the SIR ccdf of the round-robin scheduler, we use (16). When compared with the round-robin scheduler, the normalized SNR scheduler affords a higher ccdf due to the multiuser diversity effect.

C. Average Data Rate and Scheduling Gain

Proposition 4 ([5, Prop. 2]). The average data rate of normalized SNR scheduling can be written as
$$\begin{aligned}\mathbb{E}\big[\ln(1 + SIR^\star)\big] &= \int_0^\infty \frac{\mathbb{P}(SIR^\star > \theta)}{1+\theta}\, \mathrm{d}\theta \\ &= \sum_{n=0}^{\infty} f_N(n) \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k+1} \\ &\quad \times \int_0^\infty \frac{1}{(1+\theta)(1 + \sqrt{k\theta}\arctan\sqrt{k\theta})}\, \mathrm{d}\theta. \end{aligned} \tag{22}$$

The solid line in Fig. 6[5] indicates the scheduling gain, $\mathbb{E}\big[\ln(1 + SIR^\star)\big]/\mathbb{E}\big[\ln(1 + SIR)\big]$, where $\mathbb{E}\big[\ln(1+SIR)\big]$ was given in (18).

To further examine the validity of Remark 2 for integer points of λ_u/λ_b, we evaluate the approximated average data rate, which can be expressed as below with the use of (21),
$$\begin{aligned}&\mathbb{E}\big[\ln(1 + SIR^\star)\big] \\ &\quad \approx \sum_{k=1}^{\lambda_u/\lambda_b+1} \binom{\lambda_u/\lambda_b+1}{k} (-1)^{k+1} \\ &\qquad \times \int_0^\infty \frac{1}{(1+\theta)(1 + \sqrt{k\theta}\arctan\sqrt{k\theta})}\, \mathrm{d}\theta. \end{aligned} \tag{23}$$

The approximated scheduling gain (23) is also shown in Fig. 6, and we observe that the approximation (20) is sufficient for estimation of the scheduling gain.

[4]It is equivalent that N is approximated to be independent of R.

[5]The difference in the values shown in this figure and in Fig. 3 in [5] is due to low accuracy when evaluating Fig. 3 in [5].

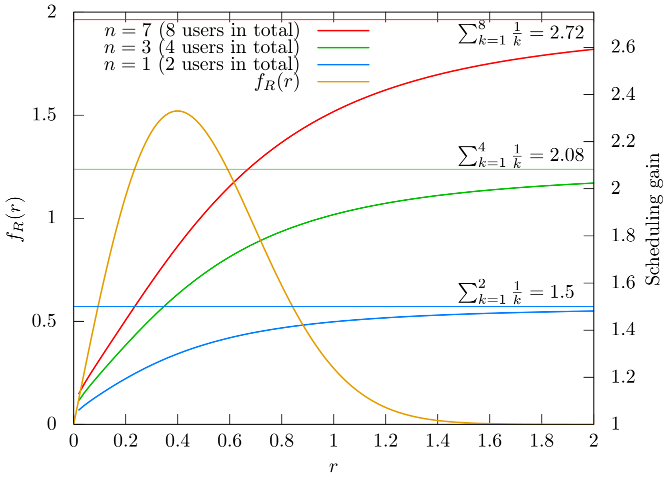

Fig. 7: Scheduling gains given r and n, (24), for $\lambda_{\mathrm{b}}=1$. Three straight lines shows the scheduling gain evaluated using (9).

Scheduling gain when the data rate is proportional to SNR (9) is also shown in Fig. 6. We can see that the scheduling gain in Poisson cellular networks is relatively small compared with that obtained with (9). To understand the reason for the absolute value of the scheduling gain being relatively small, we evaluate the scheduling gain given r and n using

$$G(r,n)=\frac{\mathbb{E}\big[\ln(1+\mathit{SIR}^{\star}_{n+1})\,\big|\,r,n\big]}{\mathbb{E}\big[\ln(1+\mathit{SIR})\,\big|\,r\big]}, \tag{24}$$

$$\begin{aligned}
&\mathbb{E}\big[\ln(1+\mathit{SIR}^{\star}_{n+1})\,\big|\,r,n\big]\\
&\quad=\int_0^\infty \frac{\mathbb{P}(\mathit{SIR}^{\star}_{n+1}>\theta\,|\,r,n)}{1+\theta}\,\mathrm{d}\theta\\
&\quad=\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k+1}\int_0^\infty \frac{\mathcal{L}_{I_r}(k\theta r^{\alpha}\,|\,r,k)}{1+\theta}\,\mathrm{d}\theta\\
&\quad=\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k+1}\\
&\qquad\times\int_0^\infty \frac{\exp(-\lambda_{\mathrm{b}}\pi r^2\sqrt{k\theta}\arctan\sqrt{k\theta})}{1+\theta}\,\mathrm{d}\theta. \qquad (25)
\end{aligned}$$

From Fig. 7, it is obvious that $G(r,n)$ increases with increase in the number of users n due to multiuser diversity gain. Further, we can see that as r increases, the scheduling gain $G(r,n)$ asymptotically approaches the value of (9). It is reasonable because the greater is the distance r, the lower is the SNR, and, thus, logarithmic rate $\ln(1+\mathit{SIR})\simeq\mathit{SIR}$ as $\mathit{SIR}\to 0$, which was assumed to derive (9).

Fig. 7 also depicts the pdf of distance, $f_R(r)$. As can be observed from the figure, the mode of the distance is around 0.4, where the scheduling gain is not very large compared to (9). Because the scheduling gain shown in Fig. 6 is obtained averaging with respect to $f_R(r)$, the scheduling gain in Poisson cellular networks is relatively small compared with that obtained with (9).

V. Conclusion

We derived the SIR ccdf, the average data rate, and the scheduling gain of the normalized SNR scheduler in both multiuser extended Poisson bipolar networks and Poisson cellular networks. In particular, the scheduling gain achieved in Poisson cellular networks was examined and it was clarified that the simplified formula under Rayleigh fading channel overestimates the scheduling gain in Poisson cellular networks.

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