SIGNAL SOURCE DETECTION USING ASYMPTOTIC BEHAVIOR OF EIGENVECTORS OF RANDOM MATRICES

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ABSTRACT

This work introduces a framework for the detection of a single source with a sensor array in the context where the noise variance and the channel between the source and the sensors are unknown at the receiver. Hypothesis test analysis is proposed. The test statistic is a function of eigenvectors and eigenvalues in the noise case. Recent results from random matrix theory are used to derive the analysis of hypothesis test, under asymptotic regime where the number of sensors and the number of observations per sensor are large but have the same order of magnitude.

Index Terms— Single source, sensor array, random matrix theory, hypothesis test

1. INTRODUCTION

The detection of unknown noisy signals or transmitter nodes is a fundamental task in various signal processing and wireless communication applications. The identification of a signal by an array of sensors constitutes a fundamental aspect of numerous wireless applications. This process holds significant relevance within the domain of cognitive radio for example, wherein a multi-sensor cognitive apparatus or a collaborative network is tasked with autonomously perceiving and understanding its surrounding environment [1]. Other examples concern the estimation of the number of high-dimensional signals in white noise using relatively few samples [2], the detection of signal in noise from finite samples using the sample eigenvalues alone [3], the detection of failures in sensor networks [4], the detection of signal in colored noise case [5], to mention just a few.

Eigenvalues based detection methods. In the spectrum sensing, which involves effectively identifying spectrum opportunities, the detection process can be cast as a simple hypothesis problem. Let $\mathbf{y}_n$ be the received signal, n is a discrete time index. So, the received signal contains samples of only noise term $\mathbf{w}_n$ or an additive combination of an information signal component $\mathbf{s}_n$ and noise :

$$\mathbf{y}_n = \begin{cases} \mathbf{w}_n & \text{under } H_0 \\ \mathbf{s}_n + \mathbf{w}_n & \text{under } H_1 \end{cases}$$

here, $\mathbf{w}_n$ is an Additive White Gaussian Noise (AWGN). In many communication models, the noise experienced by the receiver is often characterized as white (AWGN model), which implies temporal uncorrelation. Moreover, it's a common assumption to consider the noise processes of different receivers as uncorrelated. Therefore, the fundamental concept behind eigenvalue-based spectrum sensing lies in leveraging correlation within the received signal when informative signal is present. This correlation can be quantified by utilizing the sample covariance matrix. Here are some eigenvalue-based detectors [6].

- The Maximum Minimum Eigenvalue (MME) detector introduced in [7, 8] and developed in [9, 10, 11], considers as a test statistic the ratio of the largest and smallest eigenvalue of the empirical covariance matrix. Results from Random Matrix Theory (RMT) are used to develop a detection threshold.
- The generalized likelihood ratio detector [1, 12, 13] where corresponding test statistic (called Generalized Likelihood Ratio Test -GLRT) is given by the ration of the largest eigenvalue and the trace of the empirical covariance matrix. The performance of the GLRT is analyzed in the asymptotic regime when the number of sensors and the number of observations per sensor are large but have the same order of magnitude. The threshold is approximated by using asymptotic results from RMT.
- The Maximum Minus Minimum Eigenvalue (MMME) detector [14] which utilizes the largest and the smallest eigenvalues and consider the test statistic defined by the difference between these extreme eigenvalues.
- The Roy's Largest Root Test (RLRT) considers simply the largest eigenvalue as a test statistic [15, 16], which

is considered to be the Neyman-Pearson optimal under the asymptotic regime.

In this article, we propose a nouvel test statistic based on the eigenvectors and eigenvalues of the sample (empirical) covariance matrix. Results from random matrix theory concerning asymptotic behavior of the eigenvectors and eigenvalues are used to analyze the test statistic in the case where the variance of the additive noise is unknown. The rest of the paper is organized as follows. Section (2) introduces firstly some random matrix tools. The statistical test is then presented in sub-section (2.2). Simulations results are carried out in Section (3). Finally, Section (4) concludes the article.

Notations: In this paper, capital characters stand for matrices while lowercase characters stand either for scalars or vectors (bold ones). The $K \times K$ identity matrix is given by $\mathbf{I}_K$ The notation $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

2. SPECTRAL TEST STATISTIC FOR SIGNAL DETECTION

2.1. Empirical eigenvectors based distribution

Let $\mathbf{y}_n = \left[y_n^1, \ldots, y_n^K\right]^T$ be the observed $K \times 1$ complex signals, for $n = 1, \ldots, N$. In this setting, we aim to construct tests associated with the following hypothesis testing problem:

$$\mathbf{y}_n = \begin{cases} \mathbf{w}_n & \text{under } H_0 \\ \mathbf{s}_n + \mathbf{w}_n & \text{under } H_1 \end{cases}$$

where $\mathbf{w}_n = [w_{1n}, \ldots, w_{Kn}]^T$ is an additive white Gaussian noise (AWGN) with variance-covariance matrix $\sigma^2 \mathbf{I}_K$. Let $\mathbf{S}_K = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}_n \mathbf{y}_n^T$ be the empirical covariance matrix and $O_K \Delta_K O_K^*$ be the spectral decomposition of $\mathbf{S}_K$ with the eigenvalues λ_i of $\mathbf{S}_K$ arranged in nondecreasing order along the diagonal of Δ_K. The columns of the matrix O_K are the respective orthonormal eigenvectors of $\mathbf{S}_K$, and $O_K \in \mathcal{O}_K$ the $K \times K$ orthogonal group. Under null hypothesis, it is well known [17, 18, 19] that the eigenmatrix of the Wishart matrix follows the Haar distribution, representing a uniform distribution over the group of unitary matrices (or orthogonal matrices in the real case). We hypothesize that as the dimensions of the eigenmatrices O_K increase, the eigenmatrix of a large sample covariance matrix should converge towards being asymptotically Haar distributed. However, articulating the term 'asymptotically Haar distributed' presents a challenge due to the increasing dimensions of the eigenmatrices. Silverstein [17, 18] proposes the following definition. If O_K conforms to a Haar measure over the orthogonal matrices, then for any unit vector $\mathbf{x} \in \mathbb{R}^K$, the transformed vector $\mathbf{z} = O_K \mathbf{x} = (z_1, \ldots, z_K)^T$ exhibits a uniform distribution over the unit sphere $\mathcal{S}^K = \left\{\mathbf{x} \in \mathbb{R}^K; ||\mathbf{x}|| = 1\right\}$, where $||.||$ is the euclidean norm. Additionally, if $\mathbf{w} = (w_1, \ldots, w_K)^T \sim \mathcal{N}(0, I_K)$, then $\mathbf{z}$ follows the same distribution as $\mathbf{w}/||\mathbf{w}||$. Consider the stochastic process in the space $D(0,1)$ given by:

$$Y_K(t) = \sqrt{\frac{K}{2}} \sum_{k=1}^{[Kt]} \left(|z_k|^2 - \frac{1}{K} \right) \tag{1}$$

$[Kt]$ denotes the greatest integer $\leq Kt$. This stochastic process converges to a Brownian bridge $B(t)$ when K goes to infinity [17, 18]. The issue in random matrix theory (RMT) is to verify whether the same is true for general covariance matrices. Let us first give two empirical distribution functions. The first one, denoted by F^{S_K}, is called the empirical spectral distribution, defined as a counting measure based on the eigenvalues of the matrix S_K. More precisely, it counts the proportion of eigenvalues within a certain interval $[0, x]$, for given positive real x (S_K is a covariance matrix, so all eigenvalues are real positives). F^{S_K} is given by:

$$F^{S_K}(x) = \frac{1}{K} \sum_{k=1}^{K} 1_{\lambda_k \leq x} \tag{2}$$

In the seminal work of Marcenko and Pastur [20], it was proved that, when N and K go to infinity with the same rate, such that $K/N \to c > 0$, the probability measure F^{S_K} converges weakly to F^{MP}, the Marcenko-Pastur distribution with density:

$$f_{MP}(x) = \frac{\sqrt{(b-x)(x-a)}}{2\pi\sigma x c} \mathbf{1}_{[a,b]}(x)$$

where, $a = \sigma^2 \left(1 - \sqrt{c}\right)^2$ and $b = \sigma^2 \left(1 + \sqrt{c}\right)^2$.

The second important distribution function, denoted by $F_{vec}^{S_K}$, is also a counting measure, based on both eigenvalues and eigenvectors of S_K. This probability measure assigns a weight $|z_k|^2$ for eigenvalue λ_k. It is given by:

$$F_{vec}^{S_K}(x) = \sum_{k=1}^{K} |z_k|^2 1_{\{\lambda_k \leq x\}} \tag{3}$$

2.2. Test statistic for detection

For ease of application of RMT, a time transformation in $Y_K(t)$ given by (1) is made. So that, we consider the following process:

$$X_K(x) = \sqrt{\frac{K}{2}} \left(F_{vec}^{S_K}(x) - F^{S_K}(x) \right) \tag{4}$$

To examine the convergence of $X_K(x)$ we explore its linear functional, defined as

$$\begin{aligned} \hat{X}_K(g) &= \int g(x) dX_K(x) \\ &= \sqrt{\tfrac{K}{2}} \left(\sum_{k=1}^{K} |z_k|^2 g(\lambda_k) - \tfrac{1}{K} \sum_{k=1}^{K} g(\lambda_k) \right) \\ &= \sqrt{\tfrac{K}{2}} \left(\int g(x) dF_{vec}^{S_K}(x) - \int g(x) dF^{S_K}(x) \right) \end{aligned}$$

where g is a bounded continuous function. Let us now give the theorem describing the asymptotic behavior of $\hat{X}_K$.

Theorem 1 *Under the following moment assumptions:* w_{ij} *are i.i.d. centered, of variance 1 and* $\mathbb{E}|w_{ij}|^4 < \infty$*, and under the asymptotic regime* $N, K \xrightarrow[K/N\to c>0]{} \infty$*, we have:*

$$\hat{X}_K(g) \xrightarrow[K/N\to c>0]{\mathcal{D}} \mathcal{N}\left(0, \sigma_X^2\right),$$

where $\sigma_X^2 = \frac{1}{c}\left(\int g^2(x)dF^{MP}(x) - \left(\int g(x)dF^{MP}(x)\right)^2\right)$.

Proof. The proof of this result come from results from [19] for the special case $T_n = I_n$. Firstly, we have

$$F_{vec}^{S_K}(x) \xrightarrow[N\to\infty]{K/N\to c>0} F^{MP}(x)$$

and by taking $F_c(x) = F^{MP}(x)$ at this same reference, we obtain the Gaussian asymptotic behavior of $var(\hat{X}_K(g))$, with variance

$$\sigma_X^2 = \frac{1}{c}\left(\int g^2(x)dF^{MP}(x) - \left(\int g(x)dF^{MP}(x)\right)^2\right).$$

We consequently propose the following test statistic to determine the presence or not of signal.

$$T_{stat} = \sqrt{\frac{K}{2\sigma_{stat}^2}}\left(\int xdF_{vec}^{S_K}(x) - \int xdF^{S_K}(x)\right) \quad (5)$$

with $\sigma_{stat}^2 = \frac{1}{c}\int x^2 dF^{MP}(x) - \frac{1}{c}\left(\int xdF^{MP}(x)\right)^2$. Following theorem (1), under the null hypothesis, this test statistic behaves like a standard gaussian random variable.

The null hypothesis is then rejected when inequality $T_{stat} > \xi_K$ holds, where ξ_K is a certain threshold which is selected in order taht the probability of false alarme $\mathbb{P}_{H_0}(T_{stat} > \xi_K)$ does not exceed a given level α.

2.2.1. *Test statistic: unknown noise variance*

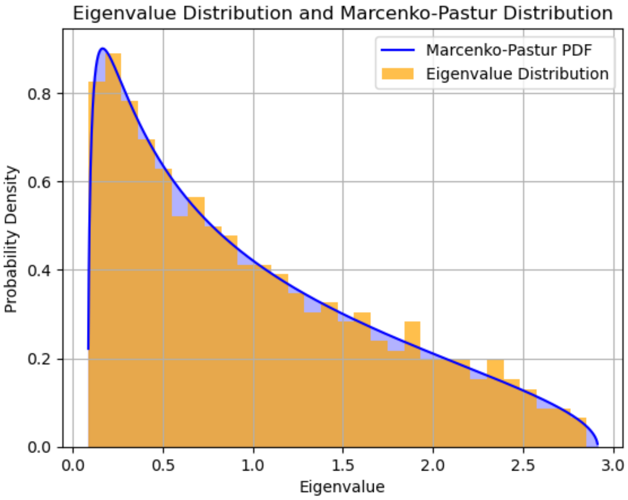
Generally, the noise variance σ^2 is unknown. We propose here to estimate this variance by using the asymptotic behavior of of the smallest eigenvalues of empirical covariance matrix $\mathbf{S}_K = \frac{1}{N}\sum_{n=1}^{N}\mathbf{y}_n\mathbf{y}_n^T$. Actually, in the scenario of pure noise (under the null hypothesis), denoted by $\mathbf{y}_n = \mathbf{w}_n$, where $\mathbf{w}_n$ is a white Gaussian noise with variance-covariance matrix $\sigma^2\mathbf{I}_K$, the seminal work of Marchenko and Pastur [20] establishes that as under the asymptotic regime $N, K \xrightarrow[K/N\to c>0]{} \infty$, all the limiting values of the sample eigenvalues λ_k fall within the support of the Marcenko-Pastur law given by the interval $[a,b] = [\sigma^2(1-\sqrt{c})^2, \sigma^2(1+\sqrt{c})^2]$ and in particular, the smallest non-null eigenvalue $\lambda_K \to \sigma^2(1-\sqrt{c})^2$, (figure 1-(a)).

Under the alternative hypothesis, the eigenvalues corresponding to the signal-information part (called the spiked eigenvalues) go to some limits outside the Marcenko-Pastur support [21], and the smallest non-null eigenvalue still goes to the limit $\sigma^2(1-\sqrt{c})^2$ (figure 1-(b)).

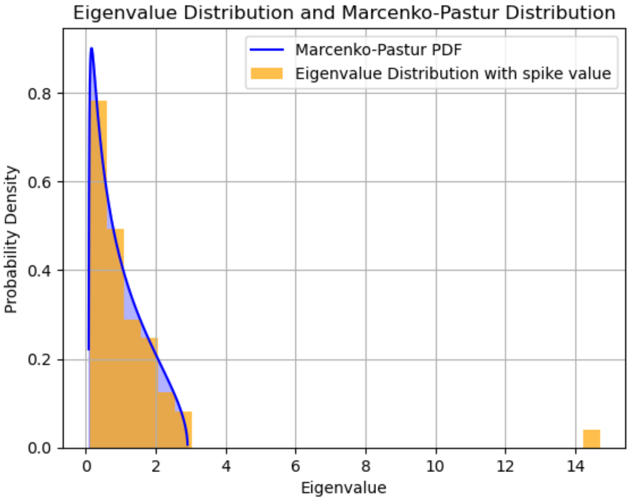
According to this, we propose to estimate σ^2 by:

$$\hat{\sigma}^2 = \lambda_{min}/(1-\sqrt{c})^2 \quad (6)$$

where, λ_{min} is the smallest non-null eigenvalue of S_K.



(a) Eigenvalues distribution in the pure noise case



(b) Eigenvalues distribution in presence of signal

Fig. 1. Empirical spectral distribution in the pure case (a) and in the presence of signal (rank one perturbation case) (b)

3. NUMERICAL RESULTS

In the following section, we analyze the performance of the proposed test. Figure 2 considers the ROC curve (Receiver Operating Characteristic) which represents the true positive rate versus the false positive rate. Here, we consider $N_1 = 50$ independent samples of $K = 50$ dimensional Gaussian white noise case and $N_2 = 50$ independent samples of $K = 50$ dimensional signal-plus-gaussian noise case. The Area Under the Curve (AUC) is equal to 0.71 which is a good performance.

Figure 3 compares the area under curve (AUC) under our test statistic versus the maximum minus minimum eigenvalue (MMME) detector by considering the same model as previously ($N_1 = 50$ independent samples of $K = 50$ dimensional Gaussian white noise case and $N_2 = 50$ independent samples of $K = 50$ dimensional signal-plus-gaussian noise case). One can observe that our model, T_{stat}, outperform the MMME one.

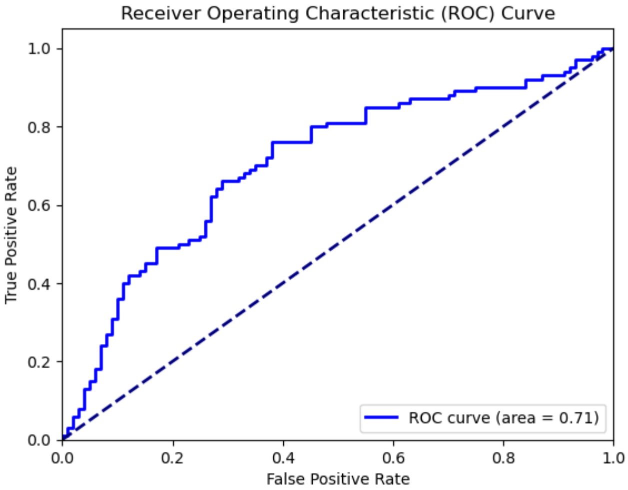


Fig. 2. ROC curve efficiency according to the asymptotic regime $N = 100$ and $K = 50$

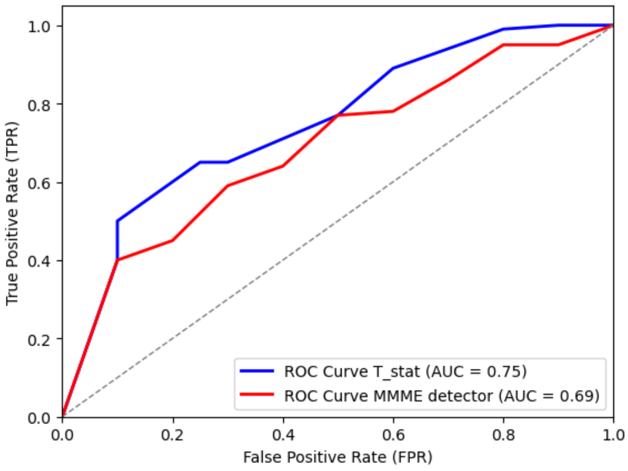


Fig. 3. Simulated ROC curve for T_{stat} and MMME detector

4. CONCLUSION

In this paper, we introduce a novel method for detecting the presence or absence of a signal. We develop a theoretical test statistic utilizing recent advancements in random matrix theory. Unlike previous approaches that rely solely on eigenvalues, our method leverages both eigenvalues and eigenvectors of the sample covariance matrix. Simulation results demonstrate the effectiveness of our approach.

5. ACKNOWLEDGMENTS

Research funding for this project was provided by MSAD team (Modélisation Stochastiques, Apprentissage et Décision) of Troyes university of technology.

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